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R. J. McAulay
E. M. Hofstetter

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LINCOLN LABORATORY

# BARANKIN BOUNDS ON PARAMETER ESTIMATION ACCURACY APPLIED TO COMMUNICATIONS AND RADAR PROBLEMS

*R. J. McAULAY*
*E. M. HOFSTETTER*

*Group 42*

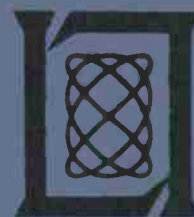TECHNICAL NOTE 1969-20

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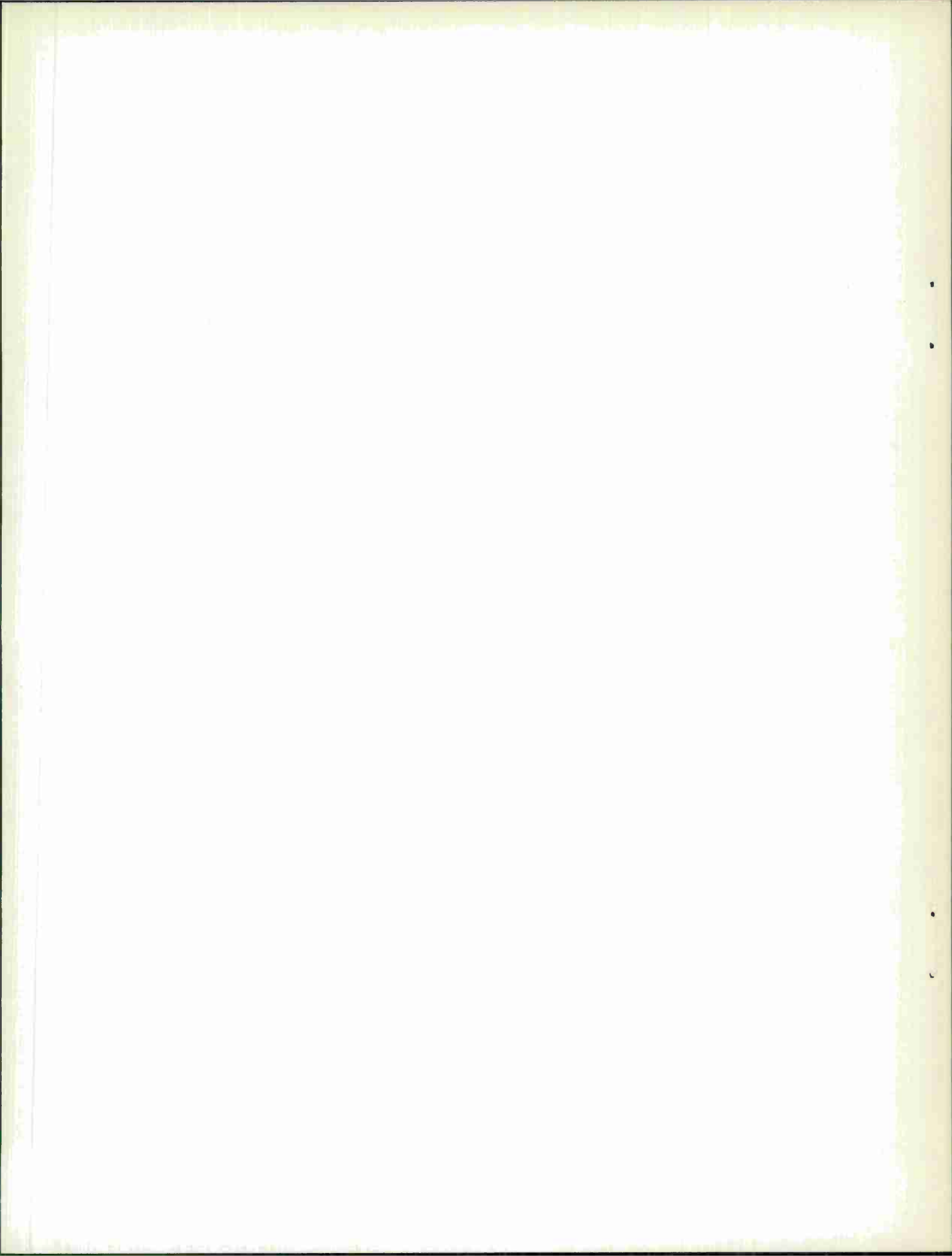LEXINGTON                                    MASSACHUSETTS

## ABSTRACT

The Schwartz Inequality is used to derive the Barankin lower bounds on the covariance matrix of unbiased estimates of a vector parameter. The bound is applied to communications and radar problems in which the unknown parameter is imbedded in a signal of known form and observed in the presence of additive white Gaussian noise. Within this context it is shown that the Barankin bound reduces to the Cramer-Rao bound when the signal-to-noise ratio (SNR) is large. However, as the SNR is reduced beyond a critical value the Barankin bound deviates radically from the Cramer-Rao bound thereby exhibiting the so-called threshold effect.

A particularly interesting signal, which has been widely used in practice to estimate the range of a target, is the linear FM waveform. The bounds were applied to this signal and within the resulting class of bounds it was possible to select one which led to a closed form expression for the lower bound on the variance of the range estimate. This expression clearly demonstrates the threshold behaviour one must expect when using a non-linear modulation system.

Tighter bounds were easily obtained but these had to be evaluated using numerical techniques. It is shown that the side-lobe structure of the linear FM compressed pulse leads to a significant increase in the variance of the estimate. For a practical linear FM pulse of 1 microsecond duration and 40 megahertz bandwidth it is shown that the radar must operate at an SNR greater than 10 dB if meaningful range estimates are to be obtained.

# I.  INTRODUCTION

An important problem in communications and radar theory is the estimation of a set of parameters $(\theta_1, \theta_2, \ldots, \theta_m)$ of a signal which has been corrupted by additive white Gaussian noise.  In particular, the received waveform is assumed to be of the form

$$r(t) = s(t; \underline{\theta}) + n(t) \qquad |t| \geqq T \qquad (1-1)$$

where $s(t; \underline{\theta})$ is a known function of t for each value of the m-row vector $\underline{\theta}$.  In the radar problem $\theta_1$ might represent an unknown time delay (target range), $\theta_2$ an unknown doppler shift (target velocity) and $\theta_3$ an unknown carrier phase angle.  The noise term n(t) represents the Gaussian white noise and is assumed to have two-sided spectral density $N_o$ watts/Hz.  It is of practical and theoretical interest to determine how well a given estimation scheme can perform with respect to estimating the unknown parameters.  In this respect, Barankin [1] has derived a general class of lower bounds on the moments of unbiased estimators. Kieffer [2] has used the Schwartz Inequality to obtain lower bounds on the variance of an unbiased estimate of a scalar-valued parameter.  Applied to the pulse-position modulation communications problem, this bound was shown to yield considerable information regarding the nonlinear modulation threshold effect [3].  In this paper, we use the Schwartz Inequality to derive lower bounds on the error covariance matrix for unbiased estimates of the vector parameter $\underline{\theta}$.  Then we specialize these results to the communications and radar problem as formulated in Equation (1-1) and apply the bound to the particular problem of estimating the range of a target using a linear FM waveform over an incoherent channel.  It is shown that the side-lobes of the corresponding compressed pulse significantly affect the variance of the estimate of the time delay.

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## II. THE BARANKIN BOUND

Let $\Omega$ be a sample space of points $\omega$ and let $P(\omega/\theta)$ be a family of probability measures on $\Omega$ indexed by the parameter $\theta$ taking values in some index set $\pi$. Assume these measures have a density function with respect to some measure $\mu$, i.e., there exists a function $p(\omega/\theta)$ such that

$$P(E/\theta) = \int_E p(\omega/\theta)d\mu(\omega)$$

for all measurable sets E.

Let $g(\cdot)$ be a real valued function defined on $\pi$ and let $\hat{g}(\cdot)$ be an unbiased estimator of $g(\theta)$, i.e., $\hat{g}(\cdot)$ is a real valued, measurable function defined on $\Omega$ with the property that

$$\int \hat{g}(\omega)p(\omega/\theta)d\mu(\omega) = g(\theta) \tag{2-1}$$

In Appendix A, we have used the Schwartz Inequality to show that the variance of the estimator $\hat{g}(\cdot)$ when $\theta$ is the true value of the unknown parameter, denoted $\sigma_\theta^2(\hat{g})$, is bounded below according to

$$\sigma_\theta^2(\hat{g}) \geq \frac{\left\{ \sum_{i=1}^{n} a_i[g(\theta_i) - g(\theta)] \right\}^2}{\int \left[ \frac{\sum_{i=1}^{n} a_i p(\omega/\theta_i)}{p(\omega/\theta)} \right]^2 p(\omega/\theta)d\mu(\omega)} \tag{2-2}$$

which is valid for all finite families $(\theta_i, a_i)$. It follows from (2-2) that,

$$\sigma_\theta^2(\hat{g}) \;\geq\; \underset{\{\theta_i, a_i\}}{\text{l.u.b.}} \;\; \frac{\left\{ \sum_{i=1}^{n} a_i[g(\theta_i) - g(\theta)] \right\}^2}{\displaystyle\int \left[ \frac{\sum_{i=1}^{n} a_i p(w/\theta_i)}{p(w/\theta)} \right]^2 p(w/\theta)\,d\mu(w)} \qquad (2\text{-}3)$$

where the l.u.b. is to be taken over all finite families of $\theta_i \in \pi$ and real $a_i$.

Equation (2-3) is the Barankin bound. Barankin has shown that this is the best possible bound in the sense that, for each $\theta$, there exists an unbiased estimator that achieves it. We will not demonstrate this here.

The preceding argument has assumed no structure on the parameter space $\pi$. We now assume that $\pi$ is the m-dimensional Euclidean space $E_m$ of vectors $\underline{\theta}$ and we will use (2-3) to derive lower bounds for the covariance matrix, $E(\hat{\underline{\theta}} - \underline{\theta})(\hat{\underline{\theta}} - \theta)'$,[*] of unbiased estimators $\hat{\underline{\theta}}$ of $\underline{\theta}$. One special case of these lower bounds will be shown to be the familiar Cramer-Rao bound.

Let $\hat{\underline{\theta}}(w)$ be an unbiased estimator of $\underline{\theta}$, i.e.,

$$E(\hat{\underline{\theta}}) = \int \hat{\underline{\theta}}(w)p(w/\underline{\theta})d\mu(w) = \underline{\theta}$$

for all $\underline{\theta} \in E_m$. If $\underline{y}$ is an arbitrary m-vector, then $\hat{g}(w) = \underline{y}'\underline{\theta}(w)$ is an unbiased estimator of $g(\underline{\theta}) = \underline{y}'\underline{\theta}$ and the bound given by Eq. (2-2) can be applied to it. This yields the inequality,

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[*]The prime denotes matrix transpose

$$\sigma_\theta^2(\hat{g}) = \underline{y}' E(\hat{\underline{\theta}} - \underline{\theta})(\hat{\underline{\theta}} - \underline{\theta})' \underline{y} \geq \frac{\left\{ \underline{y}' \sum_{i=1}^{n} a_i(\underline{\theta}_i - \underline{\theta}) \right\}^2}{\int \left[ \frac{\sum_{i=1}^{n} a_i p(\omega/\underline{\theta}_i)}{p(\omega/\underline{\theta})} \right]^2 p(\omega/\underline{\theta}) d\mu(\omega)}$$

$$(2-4)$$

and this is valid for all finite sets of vectors $\underline{\theta}_i$ in $E_m$ and all real $a_i$. For a given set of $\underline{\theta}_i$ and any set of $a_i$'s we choose, (2-4) yields a lower bound on $\sigma_\theta^2(\hat{g})$.

It is therefore reasonable to seek that set of $a_i$ which leads to the least upper bound of (2-4). The maximization is performed in detail in Appendix B where it is shown that the optimum set of $a_i$ for a fixed set of $\underline{\theta}_i$ yields the bound[*]

$$E(\hat{\underline{\theta}} - \underline{\theta})(\hat{\underline{\theta}} - \underline{\theta})' \geq \Lambda^{-1} + (\Phi - \Lambda^{-1}A) \Delta^{-1}(\Phi - \Lambda^{-1}A)' \qquad (2-5)$$

where

$$\Delta = B - A'\Lambda^{-1}A \qquad (2-6)$$

$$\Lambda = \int \frac{\partial \ln p(\omega/\underline{\theta})}{\partial \theta_i} \cdot \frac{\partial \ln p(\omega/\underline{\theta})}{\partial \theta_j} \cdot p(\omega/\underline{\theta}) d\mu(\omega) \qquad (2-7)$$
$$i, j = 1, 2, \ldots m$$

$$A = \int \frac{\partial \ln p(\omega/\underline{\theta})}{\partial \theta_i} \cdot \frac{p(\omega/\underline{\theta}_k)}{p(\omega/\underline{\theta})} \cdot p(\omega/\underline{\theta}) d\mu(\omega) \qquad (2-8)$$

$$i = 1, 2, \ldots, m; \qquad k = 1, 2, \ldots, n$$

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[*] A matrix A is said to be less than or equal to a matrix B, $A \leq B$, if B - A is non-negative definite.

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$$B = \int \frac{p(\omega/\underline{\theta}_i)}{p(\omega/\underline{\theta})} \quad \frac{p(\omega/\underline{\theta}_k)}{p(\upsilon/\underline{\theta})} \quad p(\omega/\theta) d\mu(\omega) \qquad (2-9)$$

$$i, k = 1, 2, \ldots, n$$

$$\Phi = [\underline{\theta}_1, \underline{\theta}_2, \ldots, \underline{\theta}_n] \qquad (2-10)$$

Notice that $\theta_i$, $i = 1, 2, \ldots, m$ refers to the $i^{th}$ component of the true parameter value $\underline{\theta}$, while $\underline{\theta}_i$, $i = 1, 2, \ldots, n$ refers to an arbitrary value of the parameter $\underline{\theta}$ other than its true value. These vectors can be chosen at will and the number of them, n is arbitrary. For reasons which will become clearer in the sequel, we refer to $\{\underline{\theta}_i\}_{i=1}^{n}$ as the set of test points.

If we pick n = 0, the second term on the right-hand side of Equation (2-5) vanishes and we obtain the Cramer-Rao bound.

$$E(\hat{\underline{\theta}} - \underline{\theta})(\hat{\underline{\theta}} - \underline{\theta})' \geq \Lambda^{-1}$$

For $n > 0$, we obtain other bounds that are, in general, an improvement on the Cramer-Rao bound. This follows from the fact that $\Delta$ is positive definite (Appendix B) which, in turn, implies that the matrix $(\Phi - \Lambda^{-1}A) \Delta^{-1}(\Phi - \Lambda^{-1}A)'$ is at least positive semi-definite.

## III. APPLICATION TO COMMUNICATIONS AND RADAR

It is of interest to specialize the results just obtained to the problem of estimating the parameters of a signal corrupted by additive, white Gaussian noise. More precisely, we are given a stochastic process r(t) of the form,

$$r(t) = s(t, \underline{\theta}) + n(t), \qquad |t| \lesseqgtr T \tag{3-1}$$

where $s(t, \theta)$ is a known function of t for each $\theta$ and the n(t) is zero mean white Gaussian noise process with the covariance function $E[n(t)n(s')] = N_o \delta(t-s')$. In addition we are given an unbiased estimate of $\underline{\theta}$, i.e., a measurable vector valued function, $\underline{\hat{\theta}}(\cdot)$, defined on the space of all sample functions $r(\cdot)$ and satisfying

$$E(\underline{\hat{\theta}}) = \underline{\theta}$$

We want to obtain a lower bound for $E(\underline{\hat{\theta}} - \underline{\theta})(\underline{\hat{\theta}} - \underline{\theta})'$. In this case $\Omega$ is the space of all functions $r(\cdot)$ defined on $-T \leq t \leq T$ and $p(\omega/\theta)$ is the probability density of $r(\cdot)$ relative to the measure defined by the white noise n(t) alone, i.e.,

$$p[r(\cdot)/\underline{\theta}] = \exp \left\{ \frac{1}{N_o} \int_{-T}^{T} r(t) s(t, \underline{\theta}) \, dt - \frac{1}{2N_o} \int_{-T}^{T} s^2(t, \underline{\theta}) dt \right\}$$

In order to calculate the A, B, and $\Lambda$ matrices, it is sufficient to evaluate the function

$$G(\underline{\theta}', \underline{\theta}''; \underline{\theta}) = \int \frac{p(\omega/\underline{\theta}')p(\omega/\underline{\theta}'')}{p^2(\omega/\underline{\theta})} \, p(\omega/\underline{\theta}) d\mu(\omega)$$

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For the problem at hand, this function is given by the expression

$$G(\underline{\theta}',\underline{\theta}'';\underline{\theta}) = E\left\{\frac{\exp\dfrac{1}{N_o}\displaystyle\int_{-T}^{T}r(t)s(t;\underline{\theta}')dt \;\; \exp\dfrac{1}{N_o}\displaystyle\int_{-T}^{T}r(t)s(t;\underline{\theta}'')dt}{\exp\dfrac{2}{N_o}\displaystyle\int_{-T}^{T}r(t)s(t;\underline{\theta})dt}\right\}$$

$$\cdot\left\{\frac{\exp\dfrac{-1}{N_o}\displaystyle\int_{-T}^{T}s^2(t;\underline{\theta}')dt \;\;\; \exp\dfrac{-1}{2N_o}\displaystyle\int_{-T}^{T}s^2(t;\underline{\theta}'')dt}{\exp\dfrac{-1}{N_o}\displaystyle\int_{-T}^{T}s^2(t;\underline{\theta})dt}\right\}$$

$$= E\left\{\exp\dfrac{1}{N_o}\int_{-T}^{T}n(t)\left[s(t;\underline{\theta}')-s(t;\underline{\theta})\right]dt \;\; \exp\dfrac{1}{N_o}\int_{-T}^{T}n(t)\left[s(t;\underline{\theta}'')\right.\right.$$
$$\left.\left.-s(t;\underline{\theta})\right]dt\right\}$$

$$\cdot\left\{\exp\dfrac{-1}{2N_o}\int_{-T}^{T}\left[s(t;\underline{\theta})-s(t;\theta')\right]^2 dt \;\; \exp\dfrac{-1}{2N_o}\int_{-T}^{T}\left[s(t;\underline{\theta})-s(t;\underline{\theta}'')\right]^2 dt\right\}$$

Now it is easy to verify that

$$E\left[e^x\right] = \exp\dfrac{\sigma_x^2}{2}$$

where x is a zero mean Gaussian random variable with variance $\sigma_x^2$. In our case

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$$x = \frac{1}{N_o} \int_{-T}^{T} n(t)[s(t,\underline{\theta}') - s(t,\underline{\theta})] \, dt + \frac{1}{N_o} \int_{-T}^{T} n(t)[s(t,\underline{\theta}'') - s(t,\underline{\theta})] \, dt$$

$$\sigma_x^2 = \frac{1}{N_o} \int_{-T}^{T} [s(t,\underline{\theta}') + s(t,\underline{\theta}'') - 2s(t,\underline{\theta})]^2 \, dt$$

Therefore,

$$G(\underline{\theta}',\underline{\theta}'';\underline{\theta}) = \exp\left\{\frac{1}{N_o} \int_{-T}^{T} [s(t,\underline{\theta}') - s(t,\underline{\theta})][s(t,\underline{\theta}'') - s(t,\underline{\theta})] \, dt\right\} \quad (3\text{-}2)$$

Using Eq. (3-2), it is possible to calculate explicit expressions for A, B, and $\Lambda$ as follows,

$$\Lambda = \frac{1}{N_o} \int_{-T}^{T} \frac{\partial \, s(t,\underline{\theta})}{\partial \theta_i} \frac{\partial \, s(t,\underline{\theta})}{\partial \theta_k} \, dt \quad (3\text{-}3)$$

$$i, \; k = 1, \ldots, m$$

$$A = \frac{1}{N_o} \int_{-T}^{T} \frac{\partial \, s(t,\theta)}{\partial \theta_i} [s(t,\underline{\theta}_k) - s(t,\underline{\theta})] \, dt \quad (3\text{-}4)$$

$$k = 1, 2, \ldots, n$$
$$i = 1, 2, \ldots, m$$

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$$B = \exp \left[ \frac{1}{N_o} \int_{-T}^{T} [s(t, \underline{\theta}_i) - s(t, \underline{\theta})][s(t, \underline{\theta}_k) - s(t, \underline{\theta})] \, dt \right] \quad (3\text{-}5)$$

$$i, \ k = 1, \ 2, \ \ldots, \ n$$

Using these relations the Barankin Bounds applied to the communications and radar problem becomes

$$E(\hat{\underline{\theta}}_{-}\underline{\theta})(\hat{\underline{\theta}}_{-}\underline{\theta})' \geqq \Lambda^{-1} + (\Phi - \Lambda^{-1}A)\Delta^{-1}(\Phi - \Lambda^{-1}A)' \quad (3\text{-}6)$$

where

$$\Delta = B - A'\Lambda^{-1}A \quad (3\text{-}7)$$

$$\Phi = \left[ \underline{\theta}_1, \underline{\theta}_2, \ldots, \underline{\theta}_n \right] \quad (3\text{-}8)$$

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IV.     LINEAR FM RANGE ESTIMATION ACCURACY

In the radar ranging problem a known signal

$$a(t)\cos(\omega_c t + \varphi(t)) \qquad |t| \leqq T/2 \qquad (4-1)$$

is reflected from a stationary point target, distorted in phase and observed in the presence of noise at the receiver. The received wave-form can be written as

$$a(t-\tau)\cos[\omega_c t + \varphi(t-\tau) + \gamma] + n(t) \qquad |t| \leqq T \qquad (4-2)$$

where the parameter $\tau$ represents the time delay or radar range while $\gamma$ represents a random phase shift (uniform in $[0, 2\pi]$). A typical radar receiver attempts to estimate the unknown time delay by using a likelihood signal processor. Since the radar must operate in a variety of noisy environments, ($N_o$ large and small), it is important to know how well a given signal is likely to perform for a range of signal-to-noise ratios (SNR). By applying the Barankin bounds to the above signaling problem, we can determine the lowest values of SNR above which the radar must operate to give acceptable performances. These remarks will become clearer as the analysis proceeds. From Eq. (4-2) we see that for the radar ranging problem the parameter vector is

$$\underline{\theta} = \begin{pmatrix} \tau \\ \gamma \end{pmatrix} \qquad (4-3)$$

The linear FM waveform has been the subject of considerable interest in both practice and theory. In the remainder of the paper we shall assume that the signal has the linear FM phase function so that

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$$\varphi(t) = \frac{\mu}{2} t^2 \tag{4-4}$$

where $\mu = 2\pi WT$, W is the signal bandwidth and T the time duration. In addition we shall assume that the amplitude modulation $a(t)$ rises very rapidly, but smoothly, to the constant value $\sqrt{2E/T}$ when $|t| \leqq T/2$ and that it is zero when $|t| > T/2$. Figure 1 illustrates a typical $a(t)$. Then the signal we are dealing with has the functional form

$$s(t;\underline{\theta}) = a(t-\tau)\cos\left[\omega_c t + \frac{\mu}{2}(t-\tau)^2 + \gamma\right] \tag{4-5}$$

where $\omega_c$ represents the carrier frequency in radians/sec.

In Appendix C we have carried out the detailed manipulations which lead to the Barankin bounds for the linear FM signal. It is shown that the variance of the normalized delay error is lower bounded by the following expression

$$\frac{1}{T^2} E(\hat{\tau} - \tau_o)^2 \geqq \frac{1}{E/N_o} \cdot \frac{12}{\mu^2} + \underline{\varphi}' \Delta^{-1} \underline{\varphi} \tag{4-6}$$

where

$$\varphi_i = \Delta\tau_i - \frac{12}{\mu^2} a_{1i} \tag{4-7a}$$

$$(\Delta)_{ij} = \exp\left[\frac{E}{N_o} b_{ij}\right] - \frac{E}{N_o}\left[\frac{12}{\mu^2} \underline{a}_1 \underline{a}_1' + \underline{a}_2 \underline{a}_2'\right]_{ij} \tag{4-7b}$$
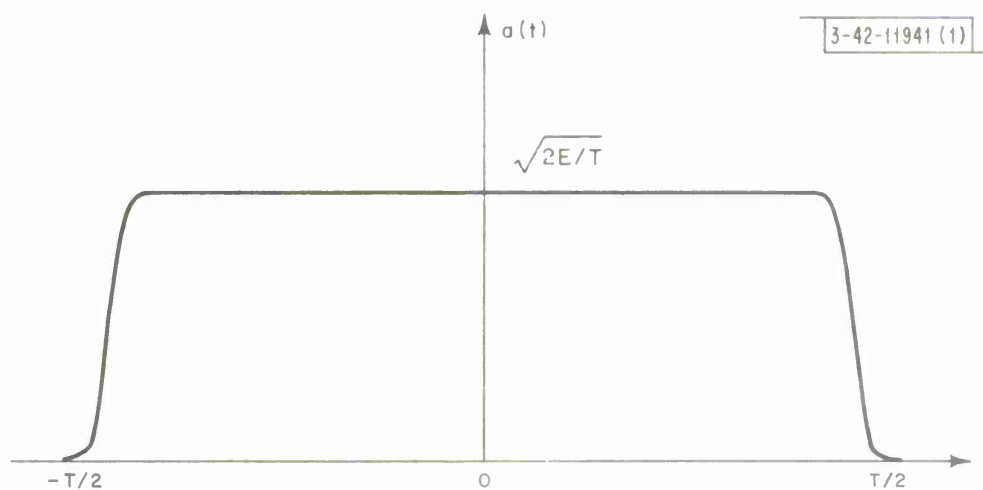
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Fig. 1. A typical "on-off" amplitude modulation.

$$b_{ij} = 1 - \frac{1}{\frac{\mu}{2}\Delta\tau_i} \cos(\Delta\gamma_i)\sin\left[\frac{\mu}{2}\Delta\tau_i(1 - |\Delta\tau_i|)\right]$$

$$- \frac{1}{\frac{\mu}{2}\Delta\tau_j} \cos(\Delta\gamma_j)\sin\left[\frac{\mu}{2}\Delta\tau_j(1 - |\Delta\tau_j|)\right] \qquad (4\text{-}7c)$$

$$+ \frac{1}{\frac{\mu}{2}(\Delta\tau_i - \Delta\tau_j)} \cos(\Delta\gamma_i - \Delta\gamma_j)\sin\left[\frac{\mu}{2}(\Delta\tau_i - \Delta\tau_j)(1 - |\Delta\tau_i - \Delta\tau_j|)\right]$$

$\underline{a}_1$ and $\underline{a}_2$ are vectors whose $j^{th}$ row is

$$a_{1j} = \left(\pm 2 - \frac{1}{\Delta\tau_j}\right)\cos(\Delta\gamma_j)\cos\left[\frac{\mu}{2}\Delta\tau_j(1 - |\Delta\tau_j|)\right]$$

$$(4\text{-}7d)$$

$$+ \frac{1}{\frac{\mu}{2}\Delta\tau_j^2} \cos(\Delta\gamma_j)\sin\left[\frac{\mu}{2}\Delta\tau_j(1 - |\Delta\tau_j|)\right]$$

where the $\pm$ signs correspond to $\Delta\tau \overset{<}{>} 0$

$$a_{2j} = \frac{1}{\frac{\mu}{2}\Delta\tau_j}\left(\pm 2 - \frac{1}{\Delta\tau_j}\right)\cos\left[\frac{\mu}{2}\Delta\tau_j(1 - |\Delta\tau_j|)\right] \qquad (4\text{-}7e)$$

$$\Delta\tau_j = (\tau_j - \tau_o)/T$$

$$(4\text{-}7f)$$

$$\Delta\gamma_j = \gamma_o - \gamma_j$$

Notice that the bounds depend only on the delay and phase differences $\tau_j - \tau_o$, $\gamma_j - \gamma_o$ and not on the true values of delay and phase $\tau_o$, $\gamma_o$. Since we know that $-T/2 \leqq \tau \leqq T/2$, then we need study only the range $-T \leqq \tau - \tau_o \leqq T$, or equivalently, $-1 \leqq \Delta\tau_j \leqq 1$. It is reasonable to assume that the phase is uniformly distributed in $[0, 2\pi]$ so that the test points, $\Delta\gamma_j$, can be chosen equally spaced in the $[0, 2\pi]$ interval. We are free to put the test points anywhere we like within the above intervals, and for each set of these points we get a lower bound on the variance of the normalized delay error. To gain some insight into how we might locate these points, let us first examine the case in which there are no phase test points. As a result the following simplified equations are obtained

$$a_{1j} = (\pm 2 - \frac{1}{\Delta\tau_j}) \cos\left[\frac{\mu}{2}\Delta\tau_j(1 - |\Delta\tau_j|)\right] \qquad (4\text{-}8a)$$

$$+ \frac{1}{\frac{\mu}{2}(\Delta\tau_j)^2} \sin\left[\frac{\mu}{2}\Delta\tau_j(1 - |\Delta\tau_j|)\right] \qquad (4\text{-}8b)$$

$$a_{2j} = 0$$

where the $\pm$ sign corresponds to $\Delta\tau_j \lessgtr 0$.

$$b_{ij} = 1 - \frac{1}{\frac{\mu}{2}\Delta\tau_i}\sin\left[\frac{\mu}{2}\Delta\tau_i(1 - |\Delta\tau_i|)\right] - \frac{1}{\frac{\mu}{2}\Delta\tau_j}\sin\left[\frac{\mu}{2}\Delta\tau_j(1 - |\Delta\tau_j|)\right]$$

$$(4\text{-}9)$$

$$+ \frac{1}{\frac{\mu}{2}(\Delta\tau_i - \Delta\tau_j)}\sin\left[\frac{\mu}{2}(\Delta\tau_i - \Delta\tau_j)(1 - |\Delta\tau_i - \Delta\tau_j|)\right]$$

It appears that the significant terms depend on the quantity $\sin\left[\frac{\mu}{2} x(1-|x|)\right]/\frac{\mu}{2}x$ which is well-known to be the response of the filter which is matched to the linear FM waveform. This filter response is used in the implementation of the maximum likelihood receiver. It is interesting that the Barankin bounds should lead to terms which depend on this receiver structure. We know from physical arguments, that the likelihood processor deviates from the Cramer-Rao performance when the noise becomes strong enough to cause the peak detector to lock on one of the sidelobe peaks of the matched filter output. Since the Barankin bounds appear to be taking this sidelobe behavior into account, we shall use this intuitive knowledge to justify locating the delay test-points at the sidelobes of the $\sin\frac{\mu}{2} x(1 - |x|)/\frac{\mu}{2} x$ function.

As an example let us choose two test points

$$\Delta\tau_1 = - \Delta\tau_2 = \delta$$

where $\delta$ is the location of the first positive sidelobe. For large time-bandwidth products we can approximate

$$\frac{\sin\frac{\mu}{2} x(1 - |x|)}{\frac{\mu}{2} x} \approx \frac{\sin\frac{\mu}{2} x}{\frac{\mu}{2} x}$$

and therefore we choose $\frac{\mu}{2} \delta = 5\pi/2$ which leads to $\delta = 2.5/WT$. Therefore the approximation is fairly good for $WT \geq 50$. Then

$$\cos\frac{\mu}{2} \delta(1 - \delta) \approx \cos\frac{\mu}{2} \delta = 0$$

and Eqs. (4-8a), (4-9) reduce to

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$$a_{11} = + WT/6.25\pi$$
$$a_{12} = - WT/6.25\pi$$
$$a_{21} = 0$$
$$a_{22} = 0$$

$$b_{11} = b_{22} = 2\left[1 - \frac{1}{\frac{\mu}{2}\delta}\sin\frac{\mu}{2}\delta(1-\delta).\right]_{\frac{\mu}{2}\delta = \frac{5\pi}{2}} \doteq 1.746$$

$$b_{12} = b_{21} = \left[1 - \frac{1}{\frac{\mu}{2}\delta}\sin\frac{\mu}{2}\delta(1-\delta) - \frac{1}{\frac{\mu}{2}(-\delta)}\sin\frac{\mu}{2}(-\delta)(1-\delta)\right.$$

$$\left. + \frac{1}{\mu\delta}\sin\mu\delta(1-2\delta)\right]_{\frac{\mu}{2}\delta = \frac{5\pi}{2}} \doteq 0.746$$

Substituting these values in Eqs. (4-7a) and (4-7b) we find that

$$\varphi_1 = -\varphi_2 = \delta - \frac{12}{(2\pi WT)^2}\frac{WT}{(6.25\pi)} = 2.485/WT$$

$$\Delta_{11} = \Delta_{22} = \exp 1.746 \frac{E}{N_o} - \frac{E}{N_o}\frac{12}{(2\pi WT)^2}\left(\frac{WT}{6.25\pi}\right)^2$$

$$= \exp 1.746 \frac{E}{N_o} - 0.00078 \frac{E}{N_o}$$

$$\approx \exp 1.746 \frac{E}{N_o}$$

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$$\Delta_{12} = \Delta_{21} = \exp 0.746 \frac{E}{N_o} + \frac{E}{N_o} \frac{12}{(2\pi WT)^2} \left( \frac{WT}{6.25\pi} \right)^2$$

$$= \exp 0.746 \frac{E}{N_o} + 0.00078 \frac{E}{N_o}$$

$$\approx \exp 0.746 \frac{E}{N_o}$$

In this case the $\Delta$-matrix is easily inverted to give

$$\Delta^{-1} = \frac{1}{\Delta_{11}^2 - \Delta_{12}^2} \begin{pmatrix} \Delta_{11} & -\Delta_{12} \\ -\Delta_{12} & \Delta_{11} \end{pmatrix}$$

and the quadratic form appearing in Eq. (4-6) reduces to

$$\underline{\varphi}^T \Delta^{-1} \underline{\varphi} = 2 \varphi_1^2 / (\Delta_{11} - \Delta_{12})$$

$$= 2 \left( \frac{2.485}{WT} \right)^2 \frac{1}{\exp 1.746 \frac{E}{N_o} - \exp 0.746 \frac{E}{N_o}}$$

$$= \frac{12.38 \exp \left( -1.746 \frac{E}{N_o} \right)}{1 - \exp \left( -\frac{E}{N_o} \right)}$$

Substituting this result into Eq. (4-6) we obtain a lower bound on the variance of the normalized delay error, namely

$$\frac{1}{T^2} \; E(\hat{\tau}-\tau_o)^2 \geqq \frac{1}{\frac{\pi^2}{3}\,(WT)^2} \quad \frac{1}{\frac{E}{N_o}} \quad T\left(\frac{E}{N_o}\right)$$

(4-10a)

where

$$T\left(\frac{E}{N_o}\right) = 1 + \frac{40.6\,\dfrac{E}{N_o}\;\exp\,(-1.746\,\dfrac{E}{N_o})}{1 - \exp\,(-\dfrac{E}{N_o})}$$

(4.10b)

When $\dfrac{E}{N_o}$ is large, $T\,(\dfrac{E}{N_o}) \Rightarrow 1$ and

$$\frac{1}{T^2} \; E\,(\hat{\tau}-\tau_o)^2 \geqq \frac{1}{\frac{\pi^2}{3}\,(WT)^2} \cdot \frac{1}{\frac{E}{N_o}}$$

which is the Cramer-Rao lower bound for the linear FM waveform. It has been shown [4] that the performance of the matched filter receiver achieves this bound when $\dfrac{E}{N_o}$ is large, but it is not known when the performance begins to deviate from it. Letting $\dfrac{E}{N_o} \Rightarrow 0$ in Equation (4-10b) we see that $T\,(\dfrac{E}{N_o}) \Rightarrow \infty$ and the so-called threshold effect has taken place. For this reason we refer to $T\,(\dfrac{E}{N_o})$ as the threshold function. The value of $\dfrac{E}{N_o}$ at which the tighter bound of Equation (4-10a) deviates from the Cramer-Rao bound is referred to as the threshold operating point. Therefore, Eq. (4-10a) gives the first analytical expression which determines the best performance the matched filter can ever achieve. Expressed in db the lower bound on the normalized delay variance is

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$$10 \log_{10} \left\{ \frac{1}{T^2} \cdot E(\hat{\tau} - \tau_o)^2 \right\} \gtreqqless - 10 \log_{10} \frac{(2\pi W T)^2}{12} \cdot \frac{E}{N_o}$$

$$+ 10 \log_{10} T\left(\frac{E}{N_o}\right)$$

$$(4-11)$$

In Figure 2 we have plotted the bound for the case $WT = 100$ and note that changing the time-bandwidth product merely shifts the curve by a constant. The figure clearly shows the threshold effect which we referred to.

One must be careful to give the correct interpretation to the bound described by Eqs. (4-10b) and (4-11). It states only that the performance of any radar which uses a linear FM waveform to estimate range can be no better than that specified by Eq. (4-11). For example if $WT = 100$ and the input SNR is 2.5 dB, then the normalized range variance must be something greater than -41.0 dB which is 7 dB larger than that predicted by the Cramer-Rao bound.

The bound can also be used to give a lower bound on the input SNR which must be used to give acceptable performance. The Barankin bound deviates from the Cramer-Rao bound when the noise at the sidelobe peaks becomes significant. This means that radar will be making large errors in range and the performance becomes unacceptable. If we locate the threshold operating point at the value of the input signal-to-noise ratio at which the Barankin bound deviates from the Cramer-Rao bound by 1/2 dB, then if only the two largest peaks are considered, Fig. 2 shows that the radar must be operated at an input SNR greater than 6 dB. This is not a useful result from a practical point of view since the radar usually operates above 13 dB, the signal-to-noise ratio at which target detection occurs with a suitably small false alarm probability. Since the curve gives only an upper bound on performance we cannot say conclusively that the practical radar
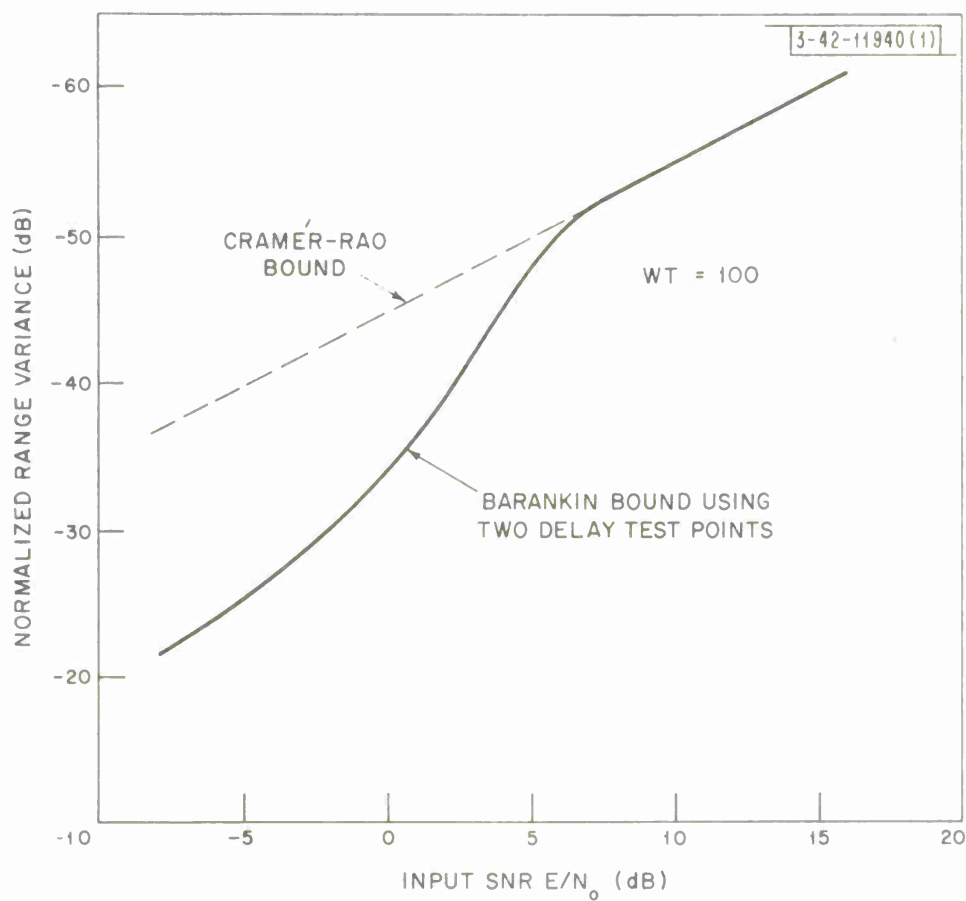
Fig. 2. Lower bound on range variance using two side-lobe test points.

operating at 13 dB input SNR will always avoid the threshold effect. However, we can obtain a tighter set of lower bounds by increasing the number of delay and phase test points. In these cases, the Δ matrix cannot easily be inverted analytically and we must resort to numerical techniques.

In Figure 3 we show the effect of taking more of the sidelobe peaks into account for a waveform having a time-bandwidth product $WT = 40$. We have plotted the lower bound for the cases where the test points are located at the first two negative peaks, then consider the additional next two positive side-lobes, etc., until all of the peaks are accounted for. When all the sidelobes, are considered a considerably tighter bound is achieved.

In the next experiment we located the delay test points at the maximum number of sidelobe peaks and selected the phase test points equally spaced in the $[0, 2\pi)$ interval. That is, if M is the number of phase test points, we set $\Delta\gamma_i = (i-1)\frac{2\pi}{(M+1)}$, $i = 1, 2, \ldots, (M+1)$. Figure 4 shows a sequence of curves which demonstrates the effect of increasing the number of phase test points. Numerical results were obtained for several time-bandwidth products up to $WT = 40$. It was found in all cases that no significant change could be observed when more than four phase test points were used, and that the lack of phase knowledge results in a loss in potential performance of about 5 dB in all cases.

In a practical radar system evaluation, one would like to relate these bounds to potential range accuracy. The range of a target, R, is related to the round trip time delay as $R = \frac{c\,\tau}{2}$

where c is the velocity of light ($3 \times 10^8$ meters/sec) and therefore, the stand-ard deviation of the range estimate, $\sigma_R$, is given by

$$\sigma_R = \left[E(\hat{R}-R_o)^2\right]^{\frac{1}{2}} = \frac{cT}{2}\left[\frac{1}{T^2}\ E(\hat{\tau}-\tau)^2\right]^{\frac{1}{2}} \tag{4-12}$$
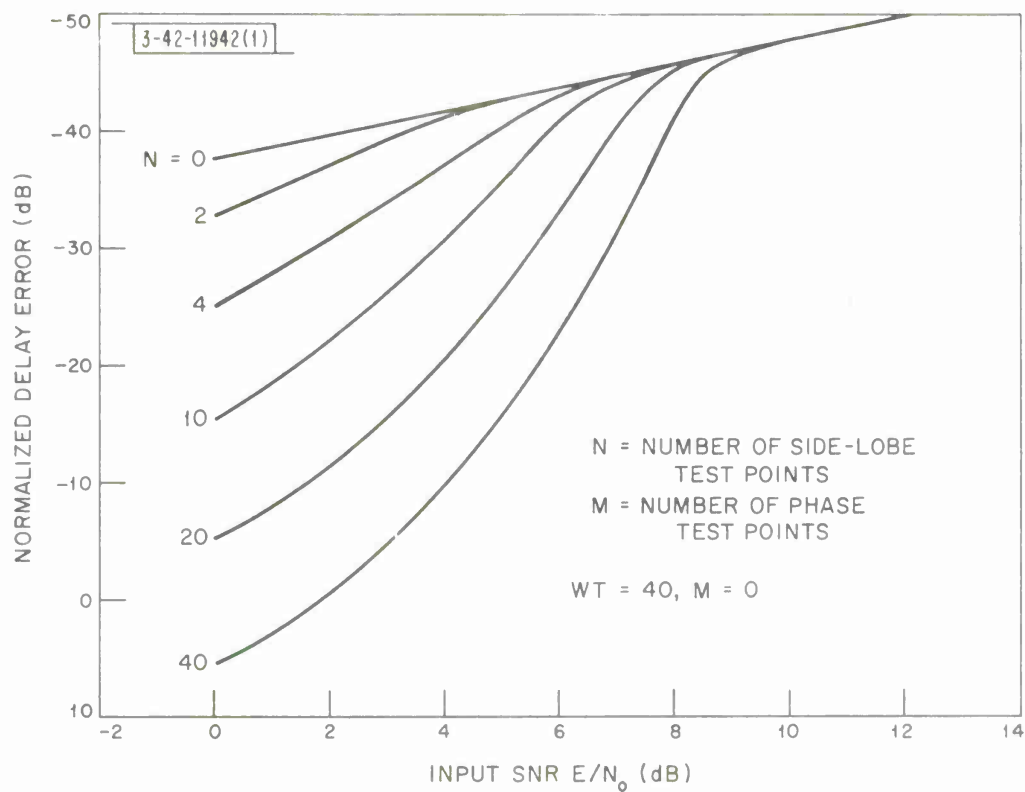
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Fig. 3.  Lower bound on range variance using additional side-lobe test points and no phase test points.

Fig. 4.  Lower bound on range variance using all the
side-lobe test points and additional phase test points.

Experiments have recently been conducted which use a linear FM waveform of 40 microseconds duration and 1 megahertz bandwidth. Therefore the $\sigma_R$ corresponding to this pulse can be computed using Equation (4-12) and the data used in plotting Figure 4. The results are shown in Figure 5.

Fig. 5. Lower bound on range error for a 1 megahertz
40 microsecond linear FM pulse.

## V.    CONCLUSIONS

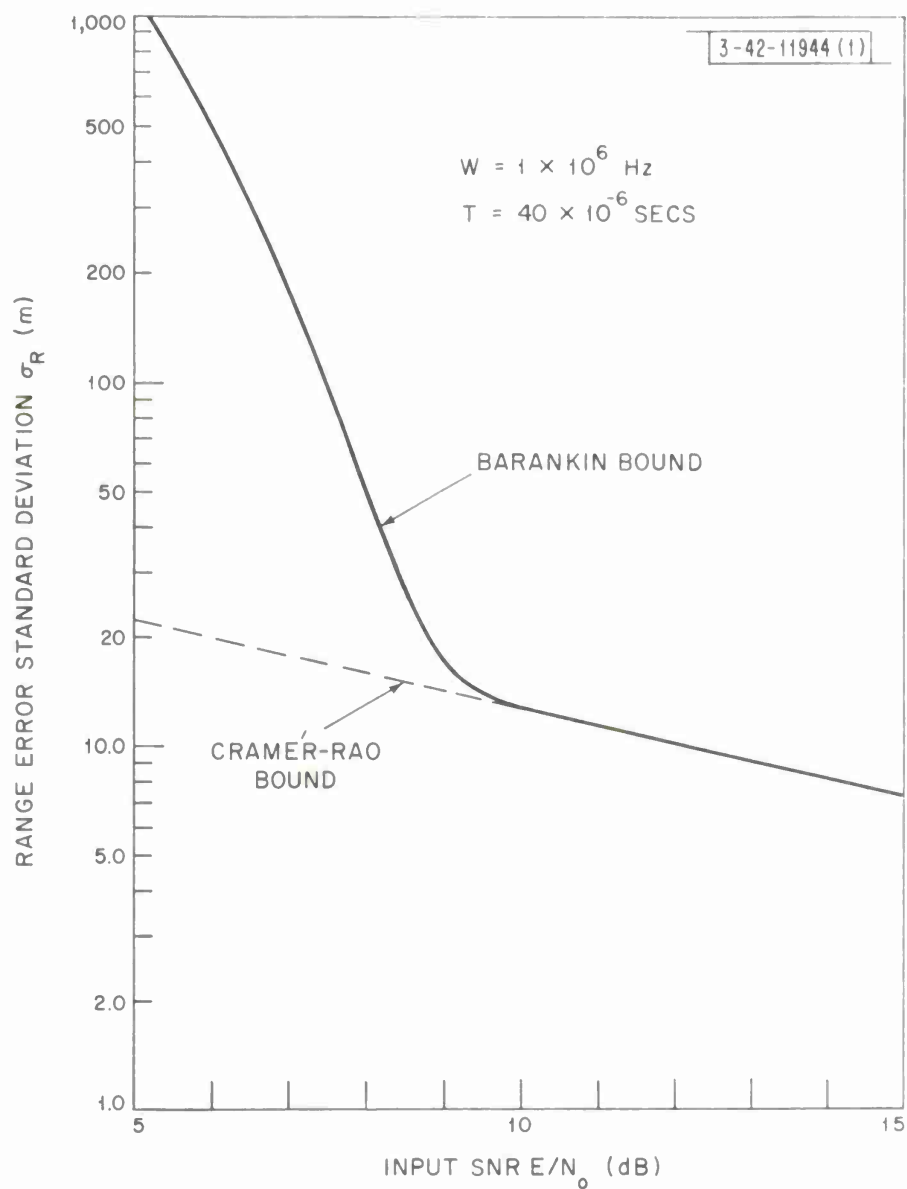A simpler derivation of the Barankin bound for the variance of unbiased estimation of vector parameters has been derived.  Under the assumption that the parameters are estimated on the basis of a signal imbedded in additive white Gaussian noise, we have reduced the bound to a form which is suitable for numerical evaluation.  It is shown that the variance depends on two terms; the first, which is the Cramer-Rao bound, involves the structure of the signal in the vicinity of the true parameter value, (fine structure), while the second term takes into account the system performance at parameter values far removed from the true values (gross structure).  It is this remarkable property of the Barankin bounds which is so useful for evaluating system performance.  Applied to the problem of estimating target range using a linear FM waveform, we see that the fine structure determines the performance in the large SNR region while the gross signal structure dominates when the SNR is small.  The bounds are extremely useful from a systems design point of view since they yield the SNR at which the transition in performance takes place.  Usually the performance is acceptable when the fine structure is predominant.  For the waveform under study, it was shown that the radar should operate above 10 dB.  Below this value, degradation in performance was rapid.  Although adequate detection capability requires an input SNR above 13 dB., there is no reason why the system could not shift  to lower SNR's when the target is being tracked.  The bound could then be used to give lower bounds on this tracking SNR.

It is interesting that the bounds derived here can be related to the side-lobes of the linear FM matched filter output.  It has long been a design criterion to use waveforms which have a low  side-lobe structure.  The Barankin bounds lead to the first analytical evidence that this is indeed a good design criterion.

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Conceptually the effect of an unknown doppler shift can easily be incorporated into the analysis although the mathematical manipulations become somewhat more complicated. However, it would be of interest to see how much the lack of knowledge of the target's velocity costs in relation to the accuracy of the range estimate.

Currently under investigation is the derivation of an upper bound for the variance of the delay error when the range is esimated using a matched filter receiver. In conjunction with the lower bounds presented here the tools for the thorough analysis of the ranging performance of a radar will then be available.

VI        <u>ACKNOWLEDGEMENT</u>

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## APPENDIX A

The following is an informal derivation of the Barankin bound. Let $\Omega$ be a sample space of points $\omega$ and let $P(\omega/\theta)$ be a family of probability measures on $\Omega$ indexed on the parameter $\theta$ taking values in some index set $\pi$. Assume these measures have a density function with respect to some measure $\mu$, i.e., there exists a function $p(\omega/\theta)$ such that

$$P(E/\theta) = \int_E p(\omega/\theta) d\mu(\omega)$$

for all measurable sets E.

Let $g(\cdot)$ be a real valued function defined on $\pi$ and let $\hat{g}(\cdot)$ be an unbiased estimator of $g(\theta)$, i.e., $\hat{g}(\cdot)$ is a real valued, measurable function defined on $\Omega$ with the property that

$$\int \hat{g}(\omega) p(\omega/\theta) d\mu(\omega) = g(\theta) \qquad (A-1)$$

We now obtain a lower bound on the variance of any such $\hat{g}(\cdot)$.

For any finite set of points $\theta_i \epsilon \pi$, $i = 1, \ldots, n$ and any set of real numbers $a_i$, $i = 1, \ldots, n$, it follows from (A-1) that

$$\int \hat{g}(\omega) \sum_{i=1}^{n} a_i p(\omega/\theta_i) d\mu(\omega) = \sum_{i=1}^{N} a_i g(\theta_i) \qquad (A-2)$$

We subtract $\sum_{i=1}^{N} a_i g(\theta)$ from both sides of (A-2) to obtain

$$\int [\hat{g}(\omega) - g(\theta)] \sum_{i=1}^{n} a_i p(\omega/\theta_i) d\mu(\omega) = \sum_{i=1}^{n} a_i [g(\theta_i) - g(\theta)] \qquad (A-3)$$

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an equation valid for $\theta_i \epsilon \pi$, $i = 1, \ldots, n$, all real $a_i$, $i = 1, \ldots, n$ and all $\theta \epsilon \pi$.

Equation (A-3) can be rewritten in the form,

$$\left\{ \int [\hat{g}(\omega) - g(\theta)] \frac{\sum\limits_{i=1}^{n} a_i p(\omega/\theta_i)}{p(\omega/\theta)} p(\omega/\theta) d\mu(\omega) \right\}^2 = \left\{ \sum\limits_{i=1}^{n} a_i [g(\theta_i) - g(\theta)] \right\}^2 \tag{A-4}$$

Applying the Schwartz inequality to the left-hand side of Eq. (A-4), we obtain

$$\left\{ \sum\limits_{i=1}^{n} \left[g(\theta_i) - g(\theta)\right] \right\}^2 \leq \left\{ \int [\hat{g}(\omega) - g(\theta)]^2 p(\omega/\theta) d\mu(\omega) \right\}$$

$$\left\{ \int \left[ \frac{\sum\limits_{i=1}^{n} a_i p(\omega/\theta_i)}{p(\omega/\theta)} \right]^2 p(\omega/\theta) d\mu(\omega) \right\} \tag{A-5}$$

The first term on the right-hand side of Eq. (A-5) is the variance of the estimator $\hat{g}(\cdot)$ when $\theta$ is the true value of the unknown parameter, $\sigma_\theta^2(\hat{g})$. We now have the inequality,

$$\sigma_\theta^2(\hat{g}) \geq \frac{\left\{ \sum\limits_{i=1}^{n} a_i [g(\theta_i) - g(\theta)] \right\}^2}{\int \left[ \frac{\sum\limits_{i=1}^{n} a_i p(\omega/\theta_i)}{p(\omega/\theta)} \right]^2 p(\omega/\theta) d\mu(\omega)} \tag{A-6}$$

valid for all finite families $\{\theta_i, a_i\}$.

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# APPENDIX B

In Section II, we showed that

$$\underline{y}'\{E(\underline{\hat{\theta}} - \underline{\theta})(\underline{\hat{\theta}} - \underline{\theta})'\}\underline{y} \geqq \frac{\{\underline{y}' \sum_{i=1}^{n} a_i(\underline{\theta}_i - \underline{\theta})\}^2}{\int \left[\sum_{i=1}^{n} a_i \frac{p(\omega/\underline{\theta}_i)}{p(\omega/\underline{\theta})}\right]^2 p(\omega/\theta)d\mu(\omega)}$$

This lower bound is valid for every set of $\underline{\theta}_i \epsilon E_m$ and all real $a_i$. We now assume a fixed set of $\underline{\theta}_i$ has been specified and then seek the set of $a_i$ which leads to the least upper bound of (B-1). To do this we now specialize some of the $\underline{\theta}_i$'s and $a_i$'s. Assume $n > m$ and define

$$\underline{\theta}_i = \underline{\theta} + \epsilon_i \underline{e}_i \ , \qquad i = 1, \ldots, m$$

$$\underline{\theta}_{m+1} = \underline{\theta}$$

where $\epsilon_i \neq 0$ is a real number and $\underline{e}_i$ is the $i^{th}$ unit vector in $E_m$, i.e.,

$$\underline{e}_i = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 1 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \leftarrow i^{th} \text{place}$$

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Further define

$$a_i = \frac{\lambda_i}{\epsilon_i}, \quad i = 1, \ldots, m$$

$$a_{m+1} = -\sum_{i=1}^{m} \frac{\lambda_i}{\epsilon_i}$$

for arbitrary real $\lambda_i$'s.

The remaining $\underline{\theta}_i$'s and $\underline{a}_i$'s, $i = m+2, \ldots, n$ are left arbitrary. With these definitions, we can write

$$\left\{ \sum_{i=1}^{n} a_i p(\omega/\underline{\theta}_i) \right\}^2 = \sum_{i,k=1}^{n} a_i a_k p(\omega/\underline{\theta}_i) p(\omega/\underline{\theta}_k)$$

$$= \sum_{i,k=1}^{m+1} a_i a_k p(\omega/\underline{\theta}_i) p(\omega/\underline{\theta}_k)$$

$$= \sum_{i,k=m+2}^{n} a_i a_k p(\omega/\underline{\theta}_i) p(\omega/\underline{\theta}_k) +$$

$$+ 2 \sum_{i=1}^{m+1} \sum_{k=m+2}^{n} a_i a_k p(\omega/\underline{\theta}_i) p(w/\underline{\theta}_k)$$

$$= \sum_{i,k=1}^{m} \lambda_i \lambda_k \quad \frac{p(\omega/\underline{\theta} + \epsilon_i \underline{e}_i) - p(\omega/\underline{\theta})}{\epsilon_i} \quad \frac{p(\omega/\underline{\theta} + \epsilon_k \underline{e}_k) - p(\omega/\underline{\theta})}{\epsilon_k}$$

$$+ 2 \sum_{i=1}^{m} \sum_{k=m+2}^{n} \lambda_i a_k \frac{p(\omega/\underline{\theta}) + \epsilon_i \underline{e}_i) - p(\omega/\underline{\theta})}{\epsilon_i} \quad p(\omega/\underline{\theta}_k)$$

$$+ \sum_{i,k=m+2}^{n} a_i a_k p(\omega/\underline{\theta}_i) p(\omega/\underline{\theta}_k)$$

It now follows that the denominator of the right-hand side of Eq. (B-1) approaches the limit

$$\int \left[ \frac{\sum_{i=1}^{n} a_i p(\omega/\underline{\theta}_i)}{p(\omega/\underline{\theta})} \right]^2 p(\omega/\underline{\theta}) d\mu(\omega)$$

$$\xrightarrow[\substack{\epsilon_i \to 0 \\ i=1,\ldots,m}]{} \sum_{i,k=1}^{m} \lambda_i \lambda_k \int \frac{\partial \ln p(\omega/\underline{\theta})}{\partial \theta_i} \quad \frac{\partial \ln p(\omega/\underline{\theta})}{\partial \theta_k} \quad p(\omega/\underline{\theta}) d\mu(\omega)$$

$$+ 2 \sum_{i=1}^{m} \sum_{k=m+2}^{n} \lambda_i a_k \int \frac{\partial \ln p(\omega/\underline{\theta})}{\partial \theta_i} \quad \frac{p(\omega/\underline{\theta}_k)}{p(\omega/\underline{\theta})} \quad p(\omega/\underline{\theta}) d\mu(\omega)$$

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$$+ \sum_{i,k=m+2}^{n} a_i a_k \int \frac{p(\omega/\underline{\theta}_i)}{p(\omega/\underline{\theta})} \; \frac{p(\omega/\underline{\theta}_k)}{p(\omega/\underline{\theta})} \; p(\omega/\underline{\theta}) d\mu(\omega)$$

where $\dfrac{\partial}{\partial \theta_i}$ denotes partial differentation with respect to the $i^{\text{th}}$ component of $\underline{\theta}$. This last expression can be written in a compact form by introducing the matrices

$$
D = \left[
\begin{array}{c|c}
\begin{array}{c} \Lambda \\ m \times m \end{array} & \begin{array}{c} A \\ m \times (n-1-m) \end{array} \\
\hline
A' & \begin{array}{c} B \\ (n-1-m) \times (n-1-m) \end{array}
\end{array}
\right]
$$

$$
\Lambda = \left[ \int \frac{\partial \ln p(\omega/\underline{\theta})}{\partial \theta_i} \cdot \frac{\partial \ln p(\omega/\underline{\theta})}{\partial \theta_k} \; p(\omega/\underline{\theta}) d\mu(\omega) \right]
$$

$$
A = \left[ \int \frac{\partial \ln p(\omega/\underline{\theta})}{\partial \theta_i} \cdot \frac{p(\omega/\underline{\theta}_{k+1})}{p(\omega/\underline{\theta})} \; p(\omega/\underline{\theta}) d\mu(\omega) \right]
$$
$$
i = 1, \ldots, m
$$
$$
k = m+1, \ldots, n-1
$$

$$
B = \left[ \int \frac{p(\omega/\underline{\theta}_{i+1})}{p(\omega/\underline{\theta})} \cdot \frac{p(\omega/\underline{\theta}_{k+1})}{p(\omega/\underline{\theta})} \; p(\omega/\underline{\theta}) d\mu(\omega) \right]
$$
$$
i, k = m+1, \ldots, n-1
$$

and the vector

$$
\underline{\ell}' = [\lambda_1, \ldots, \lambda_m, a_{m+2}, \ldots, a_n]
$$

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With this notation, the limiting form of the denominator of the right-hand side of Eq. (B-1) is $\underline{l}' D \underline{l}$. It can be shown that if $g(\underline{\theta})$ is not constant and if there exists an unbiased, finite-variance estimator of $g(\underline{\theta})$, then the denominator of Eq. (B-1) can never vanish. It follows from this fact that the matrix D is always positive definite.

With the special parameter values chosen, the numerator of the right-hand side of Eq. (B-1) becomes,

$$\left[ \underline{y}' \sum_{i=1}^{n} a_i(\underline{\theta}_i - \underline{\theta}) \right]^2 = \left[ \underline{y}' \cdot \sum_{i=1}^{m} \lambda_i \underline{e}_i + \underline{y}' \cdot \sum_{i=m+2}^{m} a_i(\underline{\theta}_i - \underline{\theta}) \right]^2 = (\underline{y}' N \underline{l})^2$$

where $N = \left[ \underline{e}_1 \cdots , \underline{e}_m, \quad (\underline{\theta}_{m+2} - \underline{\theta}), \cdots , (\underline{\theta}_n - \underline{\theta}) \right]$
$$m \times (n-1)$$

We have now obtained the following form of the Barankin bound,

$$\underline{y}' E(\hat{\underline{\theta}} - \underline{\theta})(\hat{\underline{\theta}} - \underline{\theta})' \underline{y} \geq \frac{(\underline{y}' N \underline{l})^2}{\underline{l}' D \underline{l}} \qquad (B-2)$$

valid for all (n-1)-dimensional vectors $\underline{l}$ and all choices of $\underline{\theta}_i$, i=m+2, ..., n. We now maximize the right-hand side of Eq. (B-2) with respect to $\underline{l}$. This can be done via the Schwartz inequality as follows,

$$\frac{(\underline{y}' N \underline{l})^2}{\underline{l}' D \underline{l}} = \frac{(\underline{y}' N D^{-\frac{1}{2}} D^{\frac{1}{2}} \underline{l})^2}{\underline{l}' D \underline{l}} \leq \underline{y}' N D^{-1} N' \underline{y}$$

Therefore,

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$$\underline{y}' \, E(\hat{\underline{\theta}} - \underline{\theta})(\hat{\underline{\theta}} - \underline{\theta})' \, \underline{y} \geq \underline{y}' \, N \, D^{-1} N' \underline{y}$$

or, equivalently

$$E(\hat{\underline{\theta}} - \underline{\theta})(\hat{\underline{\theta}} - \underline{\theta})' \geq N \, D^{-1} \, N' \qquad (B\text{-}3)$$

for all choices of $\underline{\theta}_i$, $i = m+2, \ldots, n$.

The relationship between the bound given by Eq. (B-3) and the Cramer-Rao bound can be brought out by means of Frobenius' formula for the inverse of a partitioned matrix. Applied to the D matrix this formula reads,

$$D^{-1} = \left[ \begin{array}{c|c} \Lambda^{-1} + \Lambda^{-1} A \, \Delta^{-1} A' \Lambda^{-1} & -\Lambda^{-1} A \Delta^{-1} \\ \hline - \Delta^{-1} A' \Lambda^{-1} & \Delta^{-1} \end{array} \right]$$

where $\Delta = b - A' \Lambda^{-1}$

Noting that $\qquad N = \left[ \begin{array}{cc} I_{m \times m} & \Phi \end{array} \right]$

where $\qquad \Phi = \left[ \underline{\theta}_{m+2} - \underline{\theta}, \ldots, \underline{\theta}_n - \underline{\theta} \right]$,

we can write

$$N \, D^{-1} N' = \Lambda^{-1} + \Lambda^{-1} A \Delta^{-1} A' \Lambda^{-1} - \Lambda^{-1} A \Delta^{-1} \Phi'$$

$$- \Phi \, \Delta^{-1} A' \, \Lambda^{-1} + \Phi \Delta^{-1} \Phi'$$

$$= \Lambda^{-1} + (\Phi - \Lambda^{-1} A) \, \Delta^{-1} (\Phi - \Lambda^{-1} A)'$$

Therefore, our bound can be written in the final form,

$$E(\hat{\underline{\theta}} - \underline{\theta})(\hat{\underline{\theta}} - \underline{\theta})' \geq \Lambda^{-1} + (\Phi - \Lambda^{-1}A)\,\Delta^{-1}(\Phi - \Lambda^{-1}A)' \qquad (B\text{-}4)$$

If we pick $n = m+1$, the second term on the right-hand side of Eq. (B-4) vanishes and we obtain the Cramer-Rao bound.

$$E(\hat{\underline{\theta}} - \underline{\theta})\,(\hat{\underline{\theta}} - \underline{\theta})' \geq \Lambda^{-1}$$

For $n > m+1$, we obtain other bounds that are, in general, an improvement on the Cramer-Rao bound. This follows from the fact that positive definiteness of D implies positive definiteness of $\Delta$, which, in turn, implies that the matrix $(\Phi - \Lambda^{-1}A)\,\Delta^{-1}\,(\Phi - \Lambda^{-1}A)'$ is at least positive semi-definite.

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## APPENDIX C

In this section we will evaluate the Barankin Bounds, Equations (3-3) to (3-8) for the linear FM waveform. We begin by computing the $\Lambda$-matrix, Eq. (3-3), where

$$\Lambda_{ij} = \frac{1}{N_o} \int_{-\infty}^{\infty} \frac{\partial s(t;\underline{\theta}_o)}{\partial \theta_i} \frac{\partial s(t;\underline{\theta}_o)}{\partial \theta_j} dt \qquad (C-1)$$
$$i, j, = 1, 2$$

We use the notation $\underline{\theta}_o$ to denote the true value of the parameter $\underline{\theta}$. The parameter vector is given by $\theta_1 = \tau$, $\theta_2 = \gamma$ and the signal, $s(t;\underline{\theta})$ is

$$s(t;\underline{\theta}) = a(t-\tau) \cos\left[\omega_c t + \frac{\mu}{2}(t-\tau)^2 + \gamma\right] \qquad (C-2)$$

so that

$$\frac{\partial s(t;\underline{\theta}_o)}{\partial \tau} = -\dot{a}(t-\tau_o) \cos\left[\omega_c t + \frac{\mu}{2}(t-\tau_o)^2 + \gamma_o\right]$$

$$+ \mu(t-\tau_o)a(t-\tau_o)\sin\left[\omega_c t + \frac{\mu}{2}(t-\tau_o)^2 + \gamma_o\right] \qquad (C-3)$$

$$\frac{\partial s(t;\underline{\theta}_o)}{\partial \gamma} = -a(t-\tau_o)\sin\left[\omega_c t + \frac{\mu}{2}(t-\tau_o)^2 + \gamma_o\right] \qquad (C-4)$$

As an example of the type of manipulations involved, we calculate the first term in detail

$$N_o \Lambda_{11} = \int_{-\infty}^{\infty} \left(\frac{\partial s(t;\underline{\theta}_o)}{\partial \tau}\right)^2 dt = \int_{-\infty}^{\infty} \left\{ \dot{a}^2(t-\tau_o) \cos^2\left[\omega_c t + \frac{\mu}{2}(t-\tau_o)^2 + \gamma_o\right] \right.$$

$$- 2\mu(t-\tau_o)a(t-\tau_o)\dot{a}(t-\tau_o) \cos\left[\omega_c t + \frac{\mu}{2}(t-\tau_o)^2 + \gamma_o\right] \sin\left[\omega_c t + \frac{\mu}{2}(t-\tau_o)^2 + \gamma_o\right]$$

$$\left. + \mu^2(t-\tau_o)^2 a^2(t-\tau_o) \sin^2\left[\omega_c t + (t-\tau_o)^2 + \gamma_o\right] \right\} dt \qquad (C-5)$$

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Using standard trigonometric identities we get terms at DC and at $2\omega_c$. The latter terms can be neglected and we obtain

$$N_o \Lambda_{11} = \frac{1}{2} \int_{-\infty}^{\infty} [\dot{a}(t-\tau_o)]^2 \, dt + \frac{u^2}{2} \int_{-\infty}^{\infty} (t-\tau_o)^2 a^2 (t-\tau_o) dt$$

$$(C-6)$$

Theoretically the modulation $a(t)$ is usually assumed to be 0 for $|t| > T/2$ and $\sqrt{2E/T}$ for $|t| \lessgtr T/2$ in which case the first integral above does not exist. However, in practice such a discontinuity cannot occur due to the bandlimitation imposed by the transmitting and receiving equipment. In this case $a(t)$ will be a "smooth" function and the contribution from the first term would be negligible compared to that of the second. We shall therefore ignore its effects. Evaluating the second integral we obtain

$$\Lambda_{11} = \frac{u^2 T^2}{12} \cdot \frac{E}{N_o}$$

$$(C-7)$$

Using similar manipulations it is easy to show that

$$\Lambda_{12} = \Lambda_{21} = 0 \tag{C-8}$$

$$\Lambda_{22} = \frac{E}{N_o} \tag{C-9}$$

Because of the simple structure of the $\Lambda$-matrix, it is easy to calculate its inverse which is

$$\Lambda^{-1} = \frac{1}{E/N_o} \begin{pmatrix} \dfrac{12}{u^2 T^2} & 0 \\ 0 & 1 \end{pmatrix} \tag{C-10}$$

Next, we compute the A-matrix defined in Eq. (3-4) as follows:

$$A_{1j} = \frac{1}{N_o} \int_{-\infty}^{\infty} \frac{\partial s(t;\underline{\theta}_o)}{\partial \tau} [s(t;\underline{\alpha}_j) - s(t;\underline{\theta}_o)]dt \qquad (C-11)$$

$$A_{2j} = \frac{1}{N_o} \int_{-\infty}^{\infty} \frac{\partial s(t;\underline{\theta}_o)}{\partial \gamma} [s(t;\underline{\alpha}_j) - s(t;\underline{\theta}_o)]dt \qquad (C-12)$$

We use $\underline{\alpha}_j$ to denote the value of the parameter $\underline{\theta}$ other than its true value. First we evaluate the term

$$\int_{-\infty}^{\infty} \frac{\partial s(t;\underline{\theta}_o)}{\partial \tau} \cdot s(t;\underline{\theta}_o)dt = \int_{-\infty}^{\infty} \left\{ -a(t-\tau_o)\dot{a}(t-\tau_o)\cos^2[\omega_c t + \frac{u}{2}(t-\tau_o)^2 + \gamma_o] \right.$$

$$\left. + u(t-\tau_o)a^2(t-\tau_o)\sin[\omega_c t + \frac{u}{2}(t-\tau_o)^2 + \gamma_o]\cos[\omega_c t + \frac{u}{2}(t-\tau_o)^2 + \gamma_o] \right\}dt$$

$$= 0 \qquad (C-13)$$

a result which follows from the fact that $a(t)$ is assumed to be an even function. Therefore

$$N_o A_{1j} = \int_{-\infty}^{\infty} \frac{\partial s(t;\underline{\theta}_o)}{\partial \tau} s(t;\underline{\alpha}_j)dt$$

$$= \int_{-\infty}^{\infty} \left\{ -a(t-\tau_j)\dot{a}(t-\tau_o)\cos\overbrace{[\omega_c t + \frac{u}{2}(t-\tau_o)^2 + \gamma_o]}^{A}\cos\overbrace{[\omega_c t + \frac{u}{2}(t-\tau_j)^2 + \gamma_j]}^{B} \right.$$

$$\left. + u(t-\tau_o)a(t-\tau_o)a(t-\tau_j)\sin\overbrace{[\omega_c t + \frac{u}{2}(t-\tau_o)^2 + \gamma_o]}^{A}\cos\overbrace{[\omega_c t + \frac{u}{2}(t-\tau_j)^2 + \gamma_j]}^{B} \right\} dt$$

$$(C-14)$$

Using the trigonometric identities

$$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$\sin A \sin B = \frac{1}{2}[\sin(A+B) + \sin(A-B)] \qquad (C-15)$$

we find that

$$N_o A_{1j} = -\frac{1}{2} \int_{-\infty}^{\infty} a(t-\tau_j)\dot{a}(t-\tau_o)\cos[ut(\tau_j-\tau_o)-\frac{u}{2}(\tau_j^2 - \tau_o^2)+(\gamma_o-\gamma_j)]dt$$

$$+\frac{u}{2}\int_{-\infty}^{\infty}(t-\tau_o)a(t-\tau_o)a(t-\tau_j)\sin[u(\tau_j-\tau_o)-\frac{u}{2}(\tau_j^2-\tau_o)^2+(\gamma_o-\gamma_j)]dt$$

$$(C-16)$$

It is instructive to evaluate the second integral first. Using the fact that $a(t)$ rises very quickly to the value $\sqrt{2E/T}$, we can use the following approximations:

Case 1: $\qquad |\tau_j - \tau_o| \geq T$

$$a(t-\tau_j)a(t-\tau_o) \approx 0 \quad \text{for all } t$$

Case 2: $\qquad \tau_j - \tau_o > 0, \quad |\tau_j - \tau_o| < T$

$$a(t-\tau_j)a(t-\tau_o) \approx \begin{cases} 2E/T & -T/2 + \tau_j < t < T/2 + \tau_o \\ 0 & \text{otherwise} \end{cases}$$

Case 3: $\qquad \tau_j - \tau_o < 0, \quad |\tau_j - \tau_o| < T$

$$a(t-\tau_j)a(t-\tau_o) \approx \begin{cases} 2E/T & -T/2 + \tau_o < t < T/2 + \tau_j \\ 0 & \text{otherwise} \end{cases}$$

If we integrate term 2, $T_2$, between the limits x and y, then we obtain

$$N_o T_2 = \frac{uE}{T}\int_x^y (t-\tau_o)\sin[ut(\tau_j-\tau_o)-\frac{u}{2}(\tau_j^2-\tau_o^2)+(\gamma_o-\gamma_j)]dt$$

$$= \frac{-E}{T(\tau_j-\tau_o)}\left\{(t-\tau_o)\cos[ut(\tau_j-\tau_o)-\frac{u}{2}(\tau_j^2-\tau_o^2)+(\gamma_o-\gamma_j)]\right.$$

$$\left. -\frac{1}{u(\tau_j-\tau_o)}\sin[ut(\tau_j-\tau_o)-\frac{u}{2}(\tau_j^2-\tau_o^2)+(\gamma_o-\gamma_j)]\right\}_x^y$$

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Using the appropriate limits for each of the three cases we see that for Case 1, $N_o T_{21} = 0$. For Case 2, we set $x = -T/2 + \tau_j$ and $y = T/2 + \tau_o$ so that

$$
N_o T_{22} = \frac{-E}{T(\tau_j - \tau_o)} \left\{ \frac{T}{2} \cos\left[ \mu\left(\frac{T}{2} + \tau_o\right)(\tau_j - \tau_o) - \frac{\mu}{2}(\tau_j^2 - \tau_o^2) + (\gamma_o - \gamma_j) \right] \right.
$$

$$
\left. - \left(-\frac{T}{2} + \tau_j - \tau_o\right) \cos\left[ \mu\left(-\frac{T}{2} + \tau_j\right)(\tau_j - \tau_o) - \frac{\mu}{2}(\tau_j^2 - \tau_o^2) + (\gamma_o - \gamma_j) \right] \right\}
$$

$$
+ \frac{E}{\mu T(\tau_j - \tau_o)^2} \left\{ \sin\left[ \mu\left(\frac{T}{2} + \tau_o\right)(\tau_j - \tau_o) - \frac{\mu}{2}(\tau_j^2 - \tau_o^2) + (\gamma_o - \gamma_j) \right] \right.
$$

$$
\left. - \sin\left[ \mu\left(-\frac{T}{2} + \tau_j\right)(\tau_j - \tau_o) - \frac{\mu}{2}(\tau_j^2 - \tau_o^2) + (\gamma_o - \gamma_j) \right] \right\} \tag{C-17}
$$

Expanding the angles which appear in this equation we obtain

$$
\mu\left(\frac{T}{2} + \tau_o\right)(\tau_j - \tau_o) - \frac{\mu}{2}(\tau_j^2 - \tau_o^2) + (\gamma_o - \gamma_j) = (\gamma_o - \gamma_j) + \frac{\mu}{2}(\tau_j - \tau_o)[T - (\tau_j - \tau_o)]
$$

$$
\mu\left(-\frac{T}{2} + \tau_o\right)(\tau_j - \tau_o) - \frac{\mu}{2}(\tau_j^2 - \tau_o^2) + (\gamma_o - \gamma_j) = (\gamma_o - \gamma_j) - \frac{\mu}{2}(\tau_j - \tau_o)[T - (\tau_j - \tau_o)]
$$

We now employ the definitions

$$
\alpha = (\gamma_o - \gamma_j)
$$

$$
\beta = \frac{\mu}{2}(\tau_j - \tau_o)[T - (\tau_j - \tau_o)]
$$

and rewrite the preceeding equations as follows

$$
N_o T_{22} = \frac{-E}{2(\tau_j - \tau_o)} [\cos(\alpha + \beta) + \cos(\alpha - \beta)] + \frac{E}{T} \cos(\alpha - \beta) + \frac{E}{\mu T(\tau_j - \tau_o)^2} [\sin(\alpha + \beta) - \sin(\alpha - \beta)]
$$

$$
\tag{C-18}
$$

Then if we use the trigonometric identities

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$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2 \cos \alpha \cos \beta$$
$$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2 \cos \alpha \sin \beta \tag{C-19}$$

we obtain

$$N_o T_{22} = \frac{-E}{(\tau_j - \tau_o)} \cos(\gamma_o - \gamma_j) \cos[\frac{\mu}{2}(\tau_j - \tau_o)[T - (\tau_j - \tau_o)]$$

$$+ \frac{E}{T} \cos\left\{(\gamma_o - \gamma_j) - \frac{\mu}{2}(\tau_j - \tau_o)[T - (\tau_j - \tau_o)]\right\}$$

$$+ \frac{2E}{\mu T(\tau_j - \tau_o)^2} \cos(\gamma_o - \gamma_j)\sin[\frac{\mu}{2}(\tau_j - \tau_o)[T - (\tau_j - \tau_o)]] \tag{C-20}$$

We repeat the same calculation for Case 3 by setting $x = -T/2 + \tau_o$ and $y = T/2 + \tau_j$ so that

$$N_o T_{23} = \frac{-E}{T(\tau_j - \tau_o)} \left\{ (\frac{T}{2} + \tau_j - \tau_o)\cos[\mu(\frac{T}{2} + \tau_j)(\tau_j - \tau_o) - \frac{\mu}{2}(\tau_j^2 - \tau_o^2) + (\gamma_o - \gamma_j)] \right.$$

$$\left. + \frac{T}{2} \cos[\frac{\mu}{2}(-\frac{T}{2} + \tau_o)(\tau_j - \tau_o) - \frac{\mu}{2}(\tau_j^2 - \tau_o^2) + (\gamma_o - \gamma_j)] \right\}$$

$$+ \frac{E}{\mu T(\tau_j - \tau_o)^2} \left\{ \sin[\mu(\frac{T}{2} + \tau_j)(\tau_j - \tau_o) - \frac{\mu}{2}(\tau_j^2 - \tau_o^2) + (\gamma_o - \gamma_j)] \right.$$

$$\left. - \sin[\mu(-\frac{T}{2} + \tau_o)(\tau_j - \tau_o) - \frac{\mu}{2}(\tau_j^2 - \tau_o^2) + (\gamma_o - \gamma_j)] \right\} \tag{C-21}$$

Expanding the angles, we get

$$\mu(\frac{T}{2} + \tau_j)(\tau_j - \tau_o) - \frac{\mu}{2}(\tau_j^2 - \tau_o^2) + (\gamma_o - \gamma_j) = (\gamma_o - \gamma_j) + \frac{\mu}{2}(\tau_j - \tau_o)[T + (\tau_j - \tau_o)]$$

$$\mu(-\frac{T}{2} + \tau_o)(\tau_j - \tau_o) - \frac{\mu}{2}(\tau_j^2 - \tau_o^2) + (\gamma_o - \gamma_j) = (\gamma_o - \gamma_j) - \frac{\mu}{2}(\tau_j - \tau_o)[T + (\tau_j - \tau_o)]$$

and now we define

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$$\alpha = (\gamma_o - \gamma_j)$$

$$\beta = \frac{u}{2}(\tau_j - \tau_o)[T + (\tau_j - \tau_o)]$$

and rewrite the preceeding equations to get

$$N_o T_{23} = \frac{-E}{2(\tau_j - \tau_o)}[\cos(\alpha+\beta) + \cos(\alpha-\beta)] - \frac{E}{T}\cos(\alpha+\beta)$$

$$+ \frac{E}{uT(\tau_j - \tau_o)^2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$

(C-22)

Making use of the identities used for Case 2, we obtain

$$N_o T_{23} = \frac{-E}{(\tau_j - \tau_o)}\cos(\gamma_o - \gamma_j)\cos[\frac{u}{2}(\tau_j - \tau_o)[T - |\tau_j - \tau_o|]]$$

$$- \frac{E}{T}\cos[(\gamma_o - \gamma_j) + \frac{u}{2}(\tau_j - \tau_o)[T - |\tau_j - \tau_o|]]$$

$$+ \frac{2E}{uT(\tau_j - \tau_o)^2}\cos(\gamma_o - \gamma_j)\sin[\frac{u}{2}(\tau_j - \tau_o)(T - |\tau_j - \tau_o|)]$$

(C-23)

There remains the computation of the first term, $T_1$, for each of the three cases. Again we make use of the fact that $a(t)$ rises very quickly to the value $\sqrt{2E/T}$, but this time we use the approximations:

Case 1: $\quad |\tau_j - \tau_o| \geq T$

$$a(t - \tau_j)\dot{a}(t - \tau_o) = 0 \quad \text{for all } t$$

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Case 2:  $\tau_j - \tau_o > 0$ , $|\tau_j - \tau_o| < T$

$$a(t-\tau_j)\dot{a}(t-\tau_o) \approx \frac{-2E}{T} \delta(t-\tau_o - T/2)$$

Case 3:  $\tau_j - \tau_o < 0$ , $|\tau_j - \tau_o| < T$

$$a(t-\tau_j)\dot{a}(t-\tau_o) \approx \frac{+2E}{T} \delta(t-\tau_o + T/2)$$

where $\delta(t)$ denotes the Dirac delta function. Now if we integrate term 1, Eq. (C-16), for Case 1 we get $N_o T_{11} = 0$, for Case 2 we get

$$N_o T_{12} = +\frac{E}{T} \cos[u(\frac{T}{2}+\tau_o)(\tau_j-\tau_o) - \frac{u}{2}(\tau_j^2 - \tau_o^2) + (\gamma_o - \gamma_j)]$$

$$= +\frac{E}{T} \cos\{(\gamma_o-\gamma_j) + \frac{u}{2}(\tau_j-\tau_o)[T - (\tau_j-\tau_o)]\} \qquad (C-24)$$

and for Case 3 we get

$$N_o T_{13} = -\frac{E}{T} \cos[u(-\frac{T}{2}+\tau_o)(\tau_j-\tau_o) - \frac{u}{2}(\tau_j^2 - \tau_o^2) + (\gamma_o - \gamma_j)]$$

$$= -\frac{E}{T} \cos\{(\gamma_o-\gamma_j) - \frac{u}{2}(\tau_j-\tau_o)[T - |\tau_j-\tau_o|]\} \qquad (C-25)$$

Finally we combine terms 1 and 2 which for Case 1 yield $A_{1j} = 0$. For Case 2 we combine Eqs. (C-20) and (C-24) to get

$$A_{1j} = -\frac{E/N_o}{(\tau_j-\tau_o)} \cos(\gamma_o-\gamma_j) \cos[\frac{u}{2}(\tau_j-\tau_o)[T-(\tau_j-\tau_o)]]$$

$$+ \frac{E/N_o}{T} \{\cos[(\gamma_o-\gamma_j) + \frac{u}{2}(\tau_j-\tau_o)[T-(\tau_j-\tau_o)]]$$

$$+ \cos[(\gamma_o-\gamma_j) - \frac{u}{2}(\tau_j-\tau_o)[T - (\tau_j-\tau_o)]]\}$$

$$+ \frac{2E/N_o}{uT(\tau_j-\tau_o)^2} \cos(\gamma_o-\gamma_j)\sin\{\frac{u}{2}(\tau_j-\tau_o)[T - (\tau_j-\tau_o)]\}$$

Using the identities in Eq. (C-19), we obtain the final result for Case 2, $\tau_j - \tau_o > 0$,

$$A_{1j} = \frac{E}{N_o} \left\{ \frac{-1}{(\tau_j - \tau_o)} \cos(\gamma_o - \gamma_j)\cos\{\frac{u}{2}(\tau_j - \tau_o)[T - (\tau_j - \tau_o)]\} \right.$$

$$+ \frac{2}{T}\cos(\gamma_o - \gamma_j)\cos\{\frac{u}{2}(\tau_j - \tau_o)[T - (\tau_j - \tau_o)]\}$$

$$\left. + \frac{2}{uT(\tau_j - \tau_o)^2}\cos(\gamma_o - \gamma_j)\sin\{\frac{u}{2}(\tau_j - \tau_o)[1 - (\tau_j - \tau_o)]\} \right\} \qquad (C-26)$$

In a similar way for Case 3, $\tau_j - \tau_o < 0$, we combine Eqs. (C-23) and (C-25) to get

$$A_{1j} = \frac{E}{N_o} \left\{ \frac{-1}{(\tau_j - \tau_o)} \cos(\gamma_o - \gamma_j)\cos\{\frac{u}{2}(\tau_j - \tau_o)[T - |\tau_j - \tau_o|]\} \right.$$

$$- \frac{2}{T}\cos(\gamma_o - \gamma_j)\cos\{\frac{u}{2}(\tau_j - \tau_o)[T - |\tau_j - \tau_o|]\}$$

$$\left. + \frac{2}{uT(\tau_j - \tau_o)^2}\cos(\gamma_o - \gamma_j)\sin\{\frac{u}{2}(\tau_j - \tau_o)[T - |\tau_j - \tau_o|]\} \right\} \qquad (C-27)$$

The second row of the A-matrix is defined by

$$A_{2j} = \frac{1}{N_o} \int_{-\infty}^{\infty} \frac{\partial s(t; \underline{\theta}_o)}{\partial \gamma} [s(t; \underline{\alpha}_j) - s(t; \underline{\theta}_o)]\, dt$$

As before we first evaluate the term

$$\int_{-\infty}^{\infty} \frac{\partial s(t; \underline{\theta}_o)}{\partial \gamma} s(t; \underline{\theta}_o)dt = - \int_{-\infty}^{\infty} a^2(t - \tau_o)\sin[w_c t + \frac{u}{2}(t - \tau_o)^2 + \gamma_o]\cos[w_c t + \frac{u}{2}(t - \tau_o)^2 + \gamma_o]\, dt$$

$$= 0$$

since we are ignoring the effects of the terms at $2\omega_c$. Therefore

$$N_o A_{2j} = \int_{-\infty}^{\infty} \frac{\partial s(t; \underline{\theta}_o)}{\partial \gamma} s(t; \underline{\alpha}_j) \, dt = - \int_{-\infty}^{\infty} a(t-\tau_o) a(t-\tau_j) \overbrace{\cos[\omega_c t + \frac{\mu}{2}(t-\tau_o)^2 + \gamma_o]}^{A}$$

$$\overbrace{\sin[\omega_c t + \frac{\mu}{2}(t-\tau_j)^2 + \gamma_j]}^{B} \, dt$$

and using the identities in Eq. (C-15), this reduces to

$$N_o A_{2j} = -\frac{1}{2} \int_{-\infty}^{\infty} a(t-\tau_o) a(t-\tau_j) \sin[\mu t(\tau_j-\tau_o) - \frac{\mu}{2}(\tau_j^2 - \tau_o^2) + (\gamma_o - \gamma_j)] \, dt$$

$$= \begin{cases} 0 & \text{for} \quad |\tau_j - \tau_o| > T \\ \frac{+E}{\mu T(\tau_j - \tau_o)} \cos[\mu t(\tau_j-\tau_o) - \frac{\mu}{2}(\tau_j^2 - \tau_o^2) + (\gamma_o - \gamma_j)] \Big|_x^y \end{cases}$$

We perform the evaluation for the case where $\tau_j - \tau_o > 0$ by setting $x = -T/2 + \tau_j$, $y = T/2 + \tau_o$ so that

$$N_o A_{2j} = + \frac{E}{\mu T(\tau_j - \tau_o)} \{ \cos[\mu(\frac{T}{2} + \tau_o)(\tau_j - \tau_o) - \frac{\mu}{2}(\tau_j^2 - \tau_o^2) + (\gamma_o - \gamma_j)]$$

$$- \cos[\mu(-\frac{T}{2} + \tau_j)(\tau_j - \tau_o) - \frac{\mu}{2}(\tau_j^2 - \tau_o^2) + (\gamma_o - \gamma_j)] \}$$

$$= \frac{-2E}{\mu T(\tau_j - \tau_o)} \sin(\gamma_o - \gamma_j) \sin\{\frac{\mu}{2}(\tau_j - \tau_o)[T - (\tau_j - \tau_o)]\} \tag{C-28}$$

a result which follows from the manipulations performed for $A_{1j}$. When $\tau_j - \tau_o < 0$ we set $x = -T/2 + \tau_o$, $y = T/2 + \tau_j$ and obtain

$$N_o A_{2j} = \frac{-2E}{\mu T(\tau_j - \tau_o)} \sin(\gamma_o - \gamma_j) \sin\{\frac{\mu}{2}(\tau_j - \tau_o)[T + (\tau_j - \tau_o)]\} \tag{C-29}$$

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Equations (C-28) and (C-29) can be combined into the single expression for $|\tau_j - \tau_o| < T$

$$A_{2j} = \frac{-2E}{\mu T(\tau_j - \tau_o)} \sin(\gamma_o - \gamma_j)\sin\{\frac{\mu}{2}(\tau_j - \tau_o)[T - |\tau_j - \tau_o|]\} \tag{C-30}$$

The B-matrix was defined in Eq. (3-5) as follows

$$B_{ij} = \exp\frac{1}{N_o} \int_{-\infty}^{\infty} [s(t;\underline{\alpha}_i) - s(t;\underline{\theta}_o)][s(t;\underline{\alpha}_j) - s(t;\underline{\theta}_o)]dt \tag{C-31}$$

First we evaluate the term

$$\int_{-\infty}^{\infty} s(t;\underline{\alpha}_i)\, s(t;\underline{\alpha}_j)dt = \int_{-\infty}^{\infty} a(t-\tau_i)a(t-\tau_j)\cos[w_c t + \frac{\mu}{2}(t-\tau_i)^2 + \gamma_i]\cos[w_c t + \frac{\mu}{2}(t-\tau_j)^2 + \gamma_j]dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} a(t-\tau_i)a(t-\tau_j)\cos[\mu t(\tau_j - \tau_i) - \frac{\mu}{2}(\tau_j^2 - \tau_i^2) + (\gamma_i - \nu_j)]dt$$

$$= \frac{E}{\mu T(\tau_j - \tau_i)} \sin[\mu t(\tau_j - \tau_i) - \frac{\mu}{2}(\tau_j^2 - \tau_i^2) + (\gamma_i - \gamma_j)]\Big|_x^y$$

We make use of manipulations which have already been performed to write the expression as

$$\int_{-\infty}^{\infty} s(t;\underline{\alpha}_i)s(t;\underline{\alpha}_j)dt = \frac{2E}{\mu T(\tau_j - \tau_o)} \cos(\gamma_i - \gamma_j)\sin\{\frac{\mu}{2}(\tau_j - \tau_i)(T - |\tau_j - \tau_i|)\}$$

The other terms in the exponent can be evaluated from this quantity to give

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$$B_{ij} = \exp \frac{E}{N_o} \left\{ 1 - \frac{2}{uT(\tau_i - \tau_o)} \cos(\gamma_i - \gamma_o) \sin[\frac{u}{2}(\tau_i - \tau_o)(T - |\tau_i - \tau_o|)] \right.$$

$$- \frac{2}{uT(\tau_j - \tau_o)} \cos(\gamma_j - \gamma_o) \sin[\frac{u}{2}(\tau_j - \tau_o)(T - |\tau_j - \tau_o|)]$$

$$\left. + \frac{2}{uT(\tau_j - \tau_i)} \cos(\gamma_j - \gamma_i) \sin[\frac{u}{2}(\tau_j - \tau_i)(T - |\tau_j - \tau_i|)] \right\} \qquad (C-32)$$

The last quantity needed in the evaluation is the $\Phi$-matrix and this is given simply by

$$\Phi = \left[ \begin{pmatrix} \tau_1 - \tau_o \\ \gamma_1 - \gamma_o \end{pmatrix} , \begin{pmatrix} \tau_2 - \tau_o \\ \gamma_2 - \gamma_o \end{pmatrix} , \quad \cdots \quad , \begin{pmatrix} \tau_n - \tau_o \\ \gamma_n - \gamma_o \end{pmatrix} \right] \qquad (C-33)$$

Some simplification can be achieved by manipulating the expression which keeps recurring in all of the formulae, namely

$$\frac{u}{2}(\tau_j - \tau_o)[T - |\tau_j - \tau_o|] = \frac{uT^2}{2} \frac{(\tau_j - \tau_o)}{T} \left[ 1 - \left| \frac{\tau_j - \tau_o}{T} \right| \right]$$

For the linear FM signal, $u = 2\pi\Delta f/T$ where $\Delta f$ represents the signal bandwidth and therefore $uT^2 = 2\pi\Delta f\ T$ where $T\ \Delta f$ is the time bandwidth product. Let us make the definition

$$\beta = 2\pi\Delta f. T \qquad (C-34)$$

so that the key angle can be written as

$$\frac{\beta}{2} \frac{\tau_j - \tau_o}{T} \left( 1 - \frac{|\tau_j - \tau_o|}{T} \right) \qquad (C-35)$$

Similarly we can write

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$$\frac{\mu T}{2}(\tau_j - \tau_o) = \frac{\mu T^2}{2} \cdot \frac{\tau_j - \tau_o}{T} = \frac{\beta}{2} \frac{\tau_j - \tau_o}{T}$$

(C-36)

and

$$- \frac{1}{\tau_j - \tau_o} \pm \frac{2}{T} = - \frac{2}{T} \quad \mp 1 + \frac{1}{\frac{2(\tau_j - \tau_o)}{T}}$$

and finally

$$\frac{\mu T}{2}(\tau_j - \tau_o)^2 = T \cdot \frac{\mu T^2}{2}\left(\frac{\tau_j - \tau_o}{T}\right)^2 = T \cdot \frac{\beta}{2}\left(\frac{\tau_j - \tau_o}{T}\right)^2$$

(C-37)

We next define the normalized delay difference

$$\Delta \tau_j = \frac{\tau_j - \tau_o}{T}$$

(C-38)

and phase difference

$$\Delta \gamma_j = \gamma_j - \gamma_o$$

(C-39)

and use the above relations to rewrite the matrices in the following more compact notation:

$$A_{ij} = \frac{1}{T} \frac{E}{N_o} a_{ij} \qquad i=1, 2; \ j=1, 2, \dots, n$$

(C-40a)

where

$$a_{1j} = \left(\pm 2 + \frac{1}{\Delta \tau_j}\right) \cos\Delta\gamma_j \cdot \cos[\frac{\beta}{2} \Delta\tau_j (1 - |\Delta\tau_j|)] + \frac{1}{\frac{\beta}{2}(\Delta\tau_j)^2} \cos\Delta\gamma_j \sin[\frac{\beta}{2}\Delta\tau_j(1 - |\Delta\tau_j|)]$$

(C-40b)

$$a_{2j} = - \frac{1}{\frac{\beta}{2}\Delta\tau_j} \sin\Delta\gamma_j \sin[\frac{\beta}{2} \Delta\tau_j (1 - |\Delta\tau_j|)]$$

(C-40c)

where the $\mp$ sign corresponds to $\Delta\tau_j \gtrless 0$.

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$$B_{ij} = \exp \frac{E}{N_o} \; b_{ij} \qquad i_{1j}=1, 2, \ldots, n$$

<div align="right">(C-41a)</div>

where

$$b_{ij} = 1 - \frac{1}{\frac{\beta}{2}\Delta\tau_i} \cos\Delta\gamma_i \cdot \sin[\frac{\beta}{2}\Delta\tau_i(1 - |\Delta\tau_i|)] - \frac{1}{\frac{\beta}{2}\Delta\tau_j} \cos\Delta\gamma_j \sin[\frac{\beta}{2}\Delta\tau_j(1 - |\Delta\tau_j|)]$$

$$+ \frac{1}{\frac{\beta}{2}(\Delta\tau_i - \Delta\tau_j)} \cos(\Delta\gamma_i - \Delta\gamma_j)\sin[\frac{\beta}{2}(\Delta\tau_i - \Delta\tau_j)(1 - |\Delta\tau_i - \Delta\tau_j|)]$$

It was stated in Eq. (3-6) that the error covariance matrix was lower bounded by

$$\Sigma = \Lambda^{-1} + (\Phi - \Lambda^{-1}A)\Delta^{-1}(\Phi - \Lambda^{-1}A)'$$

$$\Delta = B - A'\Lambda^{-1}A$$

<div align="right">(C-42)</div>

The $\Lambda^{-1}$-matrix is given by Eq.(C-10), but since $\mu^2 T^2 = \frac{1}{T^2}(\mu T^2)^2 = \frac{\beta^2}{T^2}$ we can rewrite that equation to give

$$\Lambda^{-1} = \frac{1}{E/N_o} \begin{pmatrix} \frac{12T^2}{\beta^2} & 0 \\ \\ 0 & 1 \end{pmatrix}$$

<div align="right">(C-43)</div>

Then

$$\Lambda^{-1}A = \frac{1}{E/N_o} \begin{pmatrix} \frac{12T^2}{\beta^2} & 0 \\ \\ 0 & 1 \end{pmatrix} \frac{E}{N_o} \begin{pmatrix} \frac{1}{T}a_{11}, & \frac{1}{T}a_{12}, \ldots, \frac{1}{T}a_{1n} \\ \\ a_{21}, & a_{22}, \ldots, & a_{2n} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{12T}{\beta^2}a_{11}, & \frac{12T}{\beta^2}a_{12} \cdots \frac{12T}{\beta^2}a_{1n} \\ \\ a_{21} & a_{22} & a_{2n} \end{pmatrix}$$

<div align="right">(C-44)</div>

Then

$$A'\Lambda^{-1}A = \frac{E}{N_o}\begin{pmatrix} \frac{1}{T}\,a_{11} & a_{21} \\ \frac{1}{T}\,a_{12} & a_{22} \\ \vdots & \vdots \\ \frac{1}{T}\,a_{1n} & a_{2n} \end{pmatrix}\Lambda^{-1}A$$

$$= \frac{E}{N_o}\begin{pmatrix} \frac{12}{\beta^2}a_{11}a_{11}+a_{21}a_{21}, & \frac{12}{\beta^2}a_{11}a_{12}+a_{21}a_{22}, & \cdots, & \frac{12}{\beta^2}a_{11}a_{1n}+a_{21}a_{2n} \\ \frac{12}{\beta^2}a_{12}a_{11}+a_{22}a_{21}, & \frac{12}{\beta^2}a_{12}a_{12}+a_{22}a_{22}, & \cdots, & \frac{12}{\beta^2}a_{12}a_{1n}+a_{22}a_{2n} \\ & \vdots & & \\ \frac{12}{\beta^2}a_{1n}a_{11}+a_{2n}a_{21}, & \frac{12}{\beta^2}a_{1n}a_{12}+a_{2n}a_{22}, & \cdots, & \frac{12}{\beta^2}a_{1n}a_{1n}+a_{2n}a_{2} \end{pmatrix}$$

and therefore

$$A'\Lambda^{-1}A = \frac{E}{N_o}\left(\frac{12}{\beta^2}\underline{a}_1\,\underline{a}_1' + \underline{a}_2\,\underline{a}_2'\right)$$

where $\underline{a}_1, \underline{a}_2$ are vectors whose $i^{th}$ rows are the elements $a_{1i}$, $a_{2i}$ respectively. Therefore the $\Delta$-matrix is given by

$$\Delta = B - \frac{E}{N_o}\left(\frac{12}{\beta^2}\underline{a}_1\,\underline{a}_1' + \underline{a}_2\,\underline{a}_2'\right) \tag{C-45}$$

an expression which depends only on the parameters $E/N_o$ and $\beta$.

Next we consider the matrix

$$(\Phi - \Lambda^{-1}A)_{ij} \stackrel{\Delta}{=} \varphi_{ij} \qquad i=1,2;\ j=1,2,\ldots,n$$

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From Eqs. (C-33) and (C-44) we see that

$$\varphi_{1j} = (\tau_j - \tau_o) - \frac{12T}{\beta^2} a_{1j}$$

$$\varphi_{2j} = (\gamma_j - \gamma_o) - a_{2j} \qquad \text{(C-46)}$$

It was stated that the lower bound on the error covariance matrix was given by

$$E(\hat{\underline{\theta}} - \underline{\theta}_o)(\hat{\underline{\theta}} - \underline{\theta}_o)' \geqq \Sigma$$

This means that for every vector $\underline{\ell}$,

$$\underline{\ell} \quad E(\hat{\underline{\theta}} - \underline{\theta}_o)(\hat{\underline{\theta}} - \underline{\theta}_o)' \geqq \underline{\ell}^T \Sigma \underline{\ell}$$

In particular, when $\underline{\theta} = (\tau, \gamma)'$ and when $\underline{\ell} = (1, 0)'$, we get a lower bound on the mean-squared delay error when the channel is incoherent. Applied to Eq. (C-42) using Eqs. (C-43), (C-45), and (C-46), we find that

$$E(\hat{\tau} - \tau_o)^2 \geqq \frac{1}{E/N_o} \quad \frac{T^2}{\beta^2} + \sum_{i=1}^{n} \sum_{j=1}^{n} \varphi_{1i} \, \delta_{ij} \, \varphi_{ij} \qquad \text{(C-47)}$$

where $\delta_{ij}$ represents the $ij^{\text{th}}$ element of $\Delta^{-1}$. We can normalize this result with respect to the time scale by dividing by $T^2$, which leads to the expression

$$\frac{1}{T^2} E(\hat{\tau} - \tau_o)^2 \geqq \frac{1}{E/N_o} \cdot \frac{12}{\beta^2} + \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\varphi_{1i}}{T} \delta_{ij} \frac{\varphi_{1j}}{T}$$

where now

$$\frac{1}{T} \varphi_{1i} = \frac{\tau_i - \tau_o}{T} - \frac{12}{\beta^2} a_{1i}$$

Let us set

$$\Delta\varphi_i = \frac{1}{T} \varphi_{1i} = \Delta\tau_i - \frac{12}{\beta^2} a_{1i} \qquad \text{(C-48)}$$

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$$\sigma_\tau^2 = \frac{1}{T^2} E(\hat{\tau} - \tau_o)^2$$

so that we obtain the following lower bound for the normalized delay error $\sigma_\tau^2$:

$$\sigma_\tau^2 \geq \frac{1}{E/N_o} \frac{12}{\beta^2} + \sum_{i=1}^{n} \sum_{j=1}^{n} \Delta\varphi_i \, \delta_{ij} \, \Delta\varphi_j \tag{C-49}$$

where $\Delta\varphi_i$ is given by Eq. (C-48), $\delta_{ij} = (\Delta^{-1})_{ij}$ where $\Delta$ is given by Eq. (C-45). The latter matrix involves $a_{1i}$, $a_{2i}$, $b_{ij}$ which are obtained from Eqs. (C-40) and (C-41) respectively.

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13. ABSTRACT

The Schwartz Inequality is used to derive the Barankin lower bounds on the covariance matrix of unbiased estimates of a vector parameter. The bound is applied to communications and radar problems in which the unknown parameter is imbedded in a signal of known form and observed in the presence of additive white Gaussian noise. Within this context it is shown that the Barankin bound reduces to the Cramer-Rao bound when the signal-to-noise ratio (SNR) is large. However, as the SNR is reduced beyond a critical value the Barankin bound deviates radically from the Cramer-Rao bound thereby exhibiting the so-called threshold effect.

A particularly interesting signal, which has been widely used in practice to estimate the range of a target, is the linear FM waveform. The bounds were applied to this signal and within the resulting class of bounds it was possible to select one which led to a closed form expression for the lower bound on the variance of the range estimate. This expression clearly demonstrates the threshold behaviour one must expect when using a nonlinear modulation system.

Tighter bounds were easily obtained but these had to be evaluated using numerical techniques. It is shown that the side-lobe structure of the linear FM compressed pulse leads to a significant increase in the variance of the estimate. For a practical linear FM pulse of 1 microsecond duration and 40 megahertz bandwidth it is shown that the radar must operate at an SNR greater than 10 dB if meaningful range estimates are to be obtained.

| 14. KEY WORDS | | |
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| Schwartz Inequality | Gaussian noise | linear FM |
| communications | signal-to-noise ratio | nonlinear modulation |
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